

CSE 549: Suffix Arrays

Suffix array

$T\$ = \text{abaaba}\$$

As with suffix tree,
 T is part of index

$SA(T) =$
(SA = "Suffix Array")

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

$m + 1$
integers

Suffix array of T is an array of integers in $[0, m]$ specifying the lexicographic order of $T\$$'s suffixes

Another Example Suffix Array

$s = \text{cattcat\$}$

- Idea: lexicographically sort all the suffixes.
- Store the starting indices of the suffixes in an array.

1	cattcat\$
2	attcat\$
3	ttcat\$
4	tcat\$
5	cat\$
6	at\$
7	t\$
8	\$

sort the suffixes
alphabetically



the indices just
“come along for
the ride”

8	\$
6	at\$
2	attcat\$
5	cat\$
1	cattcat\$
7	t\$
4	tcat\$
3	ttcat\$

index of suffix

suffix of s

Suffix array

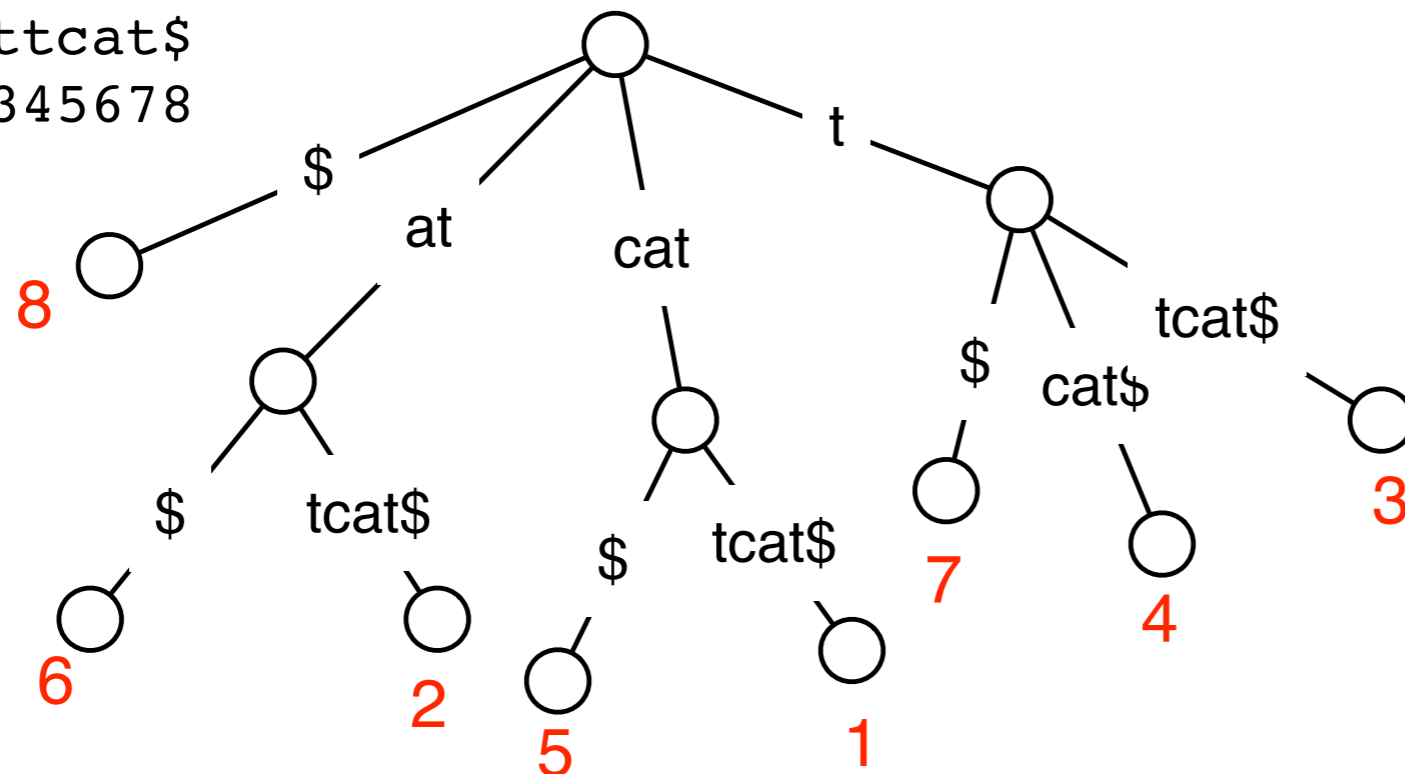
$O(m)$ space, same as suffix tree. Is constant factor smaller?

32-bit integer can distinguish characters in the human genome, so suffix array is ~12 GB, smaller than MUMmer's 47 GB suffix tree.

Relationship Between Suffix Trees & Suffix Arrays

$\Sigma = \{\$,a,c,t\}$

$s = \text{cattcat}\$$
12345678



$s = \text{cattcat}\$$

8	\$
6	at\$
2	attcat\$
5	cat\$
1	cattcat\$
7	t\$
4	tcat\$
3	ttcat\$

Red #s = starting position of the suffix ending at that leaf

Leaf labels left to right: 86251743

Edges leaving each node are sorted by label (left-to-right).

Table I. Performance Summary of the Construction Algorithms

Algorithm	Worst Case	Time	Memory
Prefix-Doubling			
MM [Manber and Myers 1993]	$O(n \log n)$	30	$8n$
LS [Larsson and Sadakane 1999]	$O(n \log n)$	3	$8n$
Recursive			
KA [Ko and Aluru 2003]	$O(n)$	2.5	$7-10n$
KS [Kärkkäinen and Sanders 2003]	$O(n)$	4.7	$10-13n$
KSPP [Kim et al. 2003]	$O(n)$	—	—
HSS [Hon et al. 2003]	$O(n)$	—	—
KJP [Kim et al. 2004]	$O(n \log \log n)$	3.5	$13-16n$
N [Na 2005]	$O(n)$	—	—
Induced Copying			
IT [Itoh and Tanaka 1999]	$O(n^2 \log n)$	6.5	$5n$
S [Seward 2000]	$O(n^2 \log n)$	3.5	$5n$
BK [Burkhardt and Kärkkäinen 2003]	$O(n \log n)$	3.5	$5-6n$
MF [Manzini and Ferragina 2004]	$O(n^2 \log n)$	1.7	$5n$
SS [Schürmann and Stoye 2005]	$O(n^2)$	1.8	$9-10n$
BB [Baron and Bresler 2005]	$O(n\sqrt{\log n})$	2.1	$18n$
M [Maniscalco and Puglisi 2007]	$O(n^2 \log n)$	1.3	$5-6n$
MP [Maniscalco and Puglisi 2006]	$O(n^2 \log n)$	1	$5-6n$
Hybrid			
IT+KA	$O(n^2 \log n)$	4.8	$5n$
BK+IT+KA	$O(n \log n)$	2.3	$5-6n$
BK+S	$O(n \log n)$	2.8	$5-6n$
Suffix Tree			
K [Kurtz 1999]	$O(n \log \sigma)$	6.3	$13-15n$

Time is relative to MP, the fastest in our experiments. Memory is given in bytes including space required for the suffix array and input string and is the average space required in our experiments. Algorithms HSS and N are included, even though to our knowledge they have not been implemented. The time for algorithm MM is estimated from experiments in Larsson and Sadakane [1999].

Suffix array: querying

Is P a substring of T ?

1. For P to be a substring, it must be a prefix of ≥ 1 of T 's suffixes
2. Suffixes sharing a prefix are consecutive in the suffix array

Use binary search

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

Suffix array: binary search

Python has `bisect` module for binary search

`bisect.bisect_left(a, x)`: Leftmost offset where we can insert `x` into `a` to maintain sorted order. `a` is already sorted!

`bisect.bisect_right(a, x)`: Like `bisect_left`, but returning *rightmost* instead of leftmost offset

```
from bisect import bisect_left, bisect_right

a = [1, 2, 3, 3, 3, 4, 5]
print(bisect_left(a, 3), bisect_right(a, 3)) # output: (2, 5)

a = [2, 4, 6, 8, 10]
print(bisect_left(a, 5), bisect_right(a, 5)) # output: (2, 2)
```

Python example: <http://nbviewer.ipython.org/6753277>

Suffix array: binary search

We can straightforwardly use binary search to find a range of elements in a sorted list that *equal* some query:

```
from bisect import bisect_left, bisect_right

strls = ['a', 'awkward', 'awl', 'awls', 'axe', 'axes', 'bee']

# Get range of elements that equal query string 'awl'
st, en = bisect_left(strls, 'awl'), bisect_right(strls, 'awl')

print(st, en) # output: (2, 3)
```

Python example: <http://nbviewer.ipython.org/6753277>

Suffix array: binary search

Can also use binary search to find a range of elements in a sorted list with some query as a *prefix*:

```
from bisect import bisect_left, bisect_right

strls = ['a', 'awkward', 'awl', 'awls', 'axe', 'axes', 'bee']

# Get range of elements with 'aw' as a prefix
st, en = bisect_left(strls, 'aw'), bisect_left(strls, 'ax')

print(st, en) # output: (1, 4)
```

Python example: <http://nbviewer.ipython.org/6753277>

Suffix array: binary search

We can do the same thing for a sorted list of suffixes:

```
from bisect import bisect_left, bisect_right

t = 'abaaba$'
suffixes = sorted([t[i:] for i in xrange(len(t))])

st, en = bisect_left(suffixes, 'aba'),
        bisect_left(suffixes, 'abb')

print(st, en) # output: (3, 5)
```

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

Python example: <http://nbviewer.ipython.org/6753277>

Suffix array: querying

Is P a substring of T ?

Do binary search, check whether P is a prefix of the suffix there

How many times does P occur in T ?

Two binary searches yield the range of suffixes with P as prefix; size of range equals # times P occurs in T

Worst-case time bound?

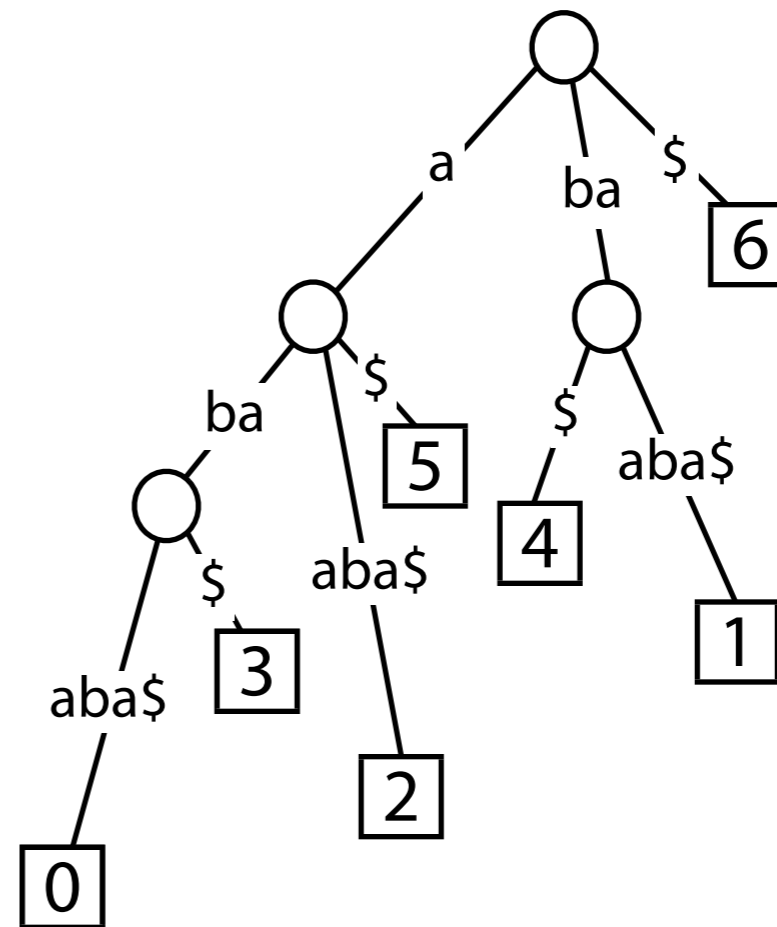
$O(\log_2 m)$ bisections, $O(n)$ comparisons per bisection, so $O(n \log m)$

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

Suffix array: querying

Contrast suffix array: $O(n \log m)$ with suffix tree: $O(n)$

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$



But we can improve bound for suffix array...

Suffix array: querying

Consider further: binary search for suffixes with P as a prefix

Assume there's no $\$$ in P . So P can't be equal to a suffix.

Initialize $l = 0$, $c = \text{floor}(m/2)$ and $r = m$ (just past last elt of SA)

\uparrow \uparrow \uparrow
"left" "center" "right"

Notation: We'll use $\text{SA}[l]$ to refer to the suffix corresponding to suffix-array element l . We could write $\pi[\text{SA}[l]:]$, but that's too verbose.

Throughout the search, invariant is maintained:

$$\text{SA}[l] < P < \text{SA}[r]$$

Suffix array: querying

Throughout search, invariant is maintained:

$$\mathbf{SA}[l] < P < \mathbf{SA}[r]$$

What do we do at each iteration?

Let $c = \text{floor}((r + l) / 2)$

If $P < \mathbf{SA}[c]$, either stop or let $r = c$ and iterate

If $P > \mathbf{SA}[c]$, either stop or let $l = c$ and iterate

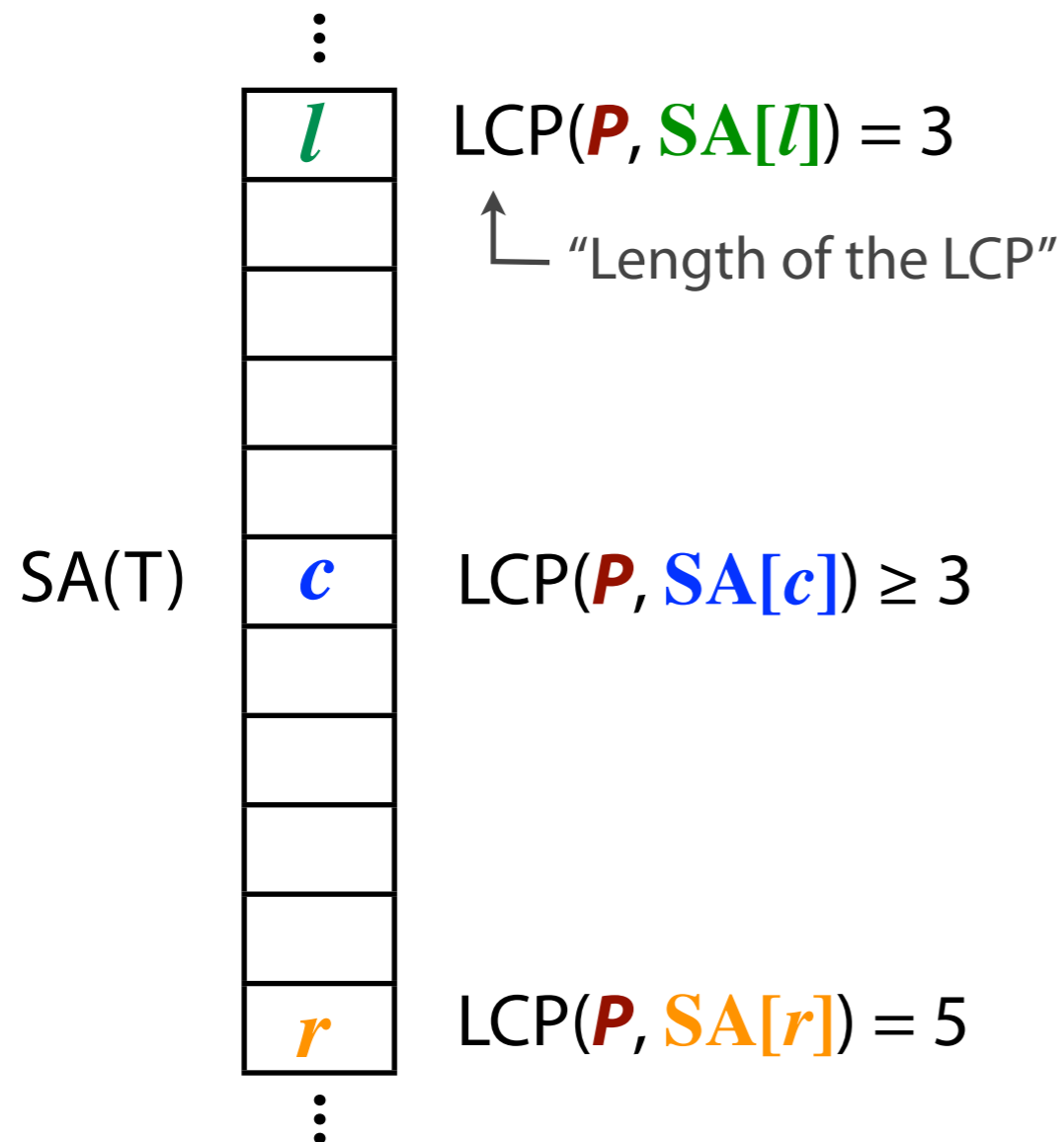
When to stop?

$P < \mathbf{SA}[c]$ and $c = l + 1$ - answer is c

$P > \mathbf{SA}[c]$ and $c = r - 1$ - answer is r

Suffix array: querying

Say we're comparing P to $SA[c]$ and we've already compared P to $SA[l]$ and $SA[r]$ in previous iterations.



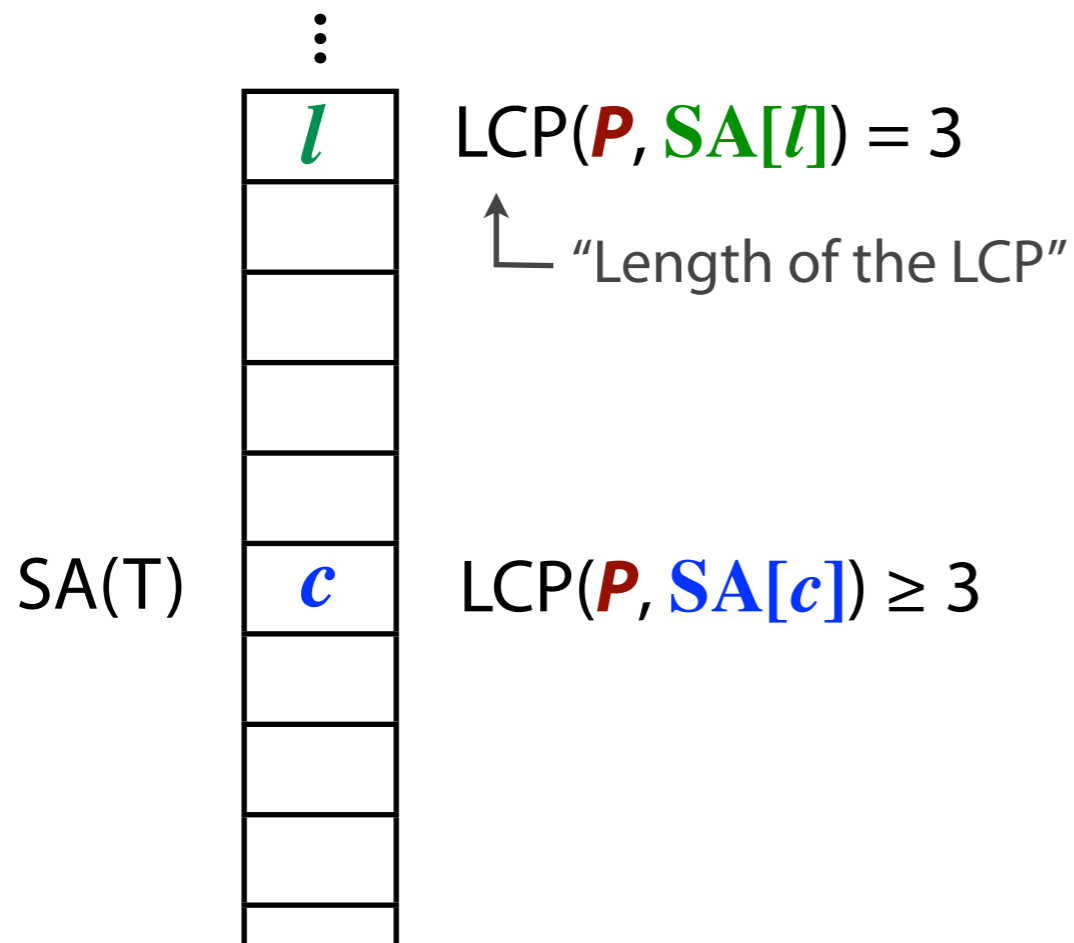
More generally:

$$\text{LCP}(P, SA[c]) \geq \min(\text{LCP}(P, SA[l]), \text{LCP}(P, SA[r]))$$

We can skip character comparisons

Suffix array: querying

Say we're comparing P to $SA[c]$ and we've already compared P to $SA[l]$ and $SA[r]$ in previous iterations.



More generally:

$$LCP(P, SA[c]) \geq \min(LCP(P, SA[l]), LCP(P, SA[r]))$$

We can skip character comparisons

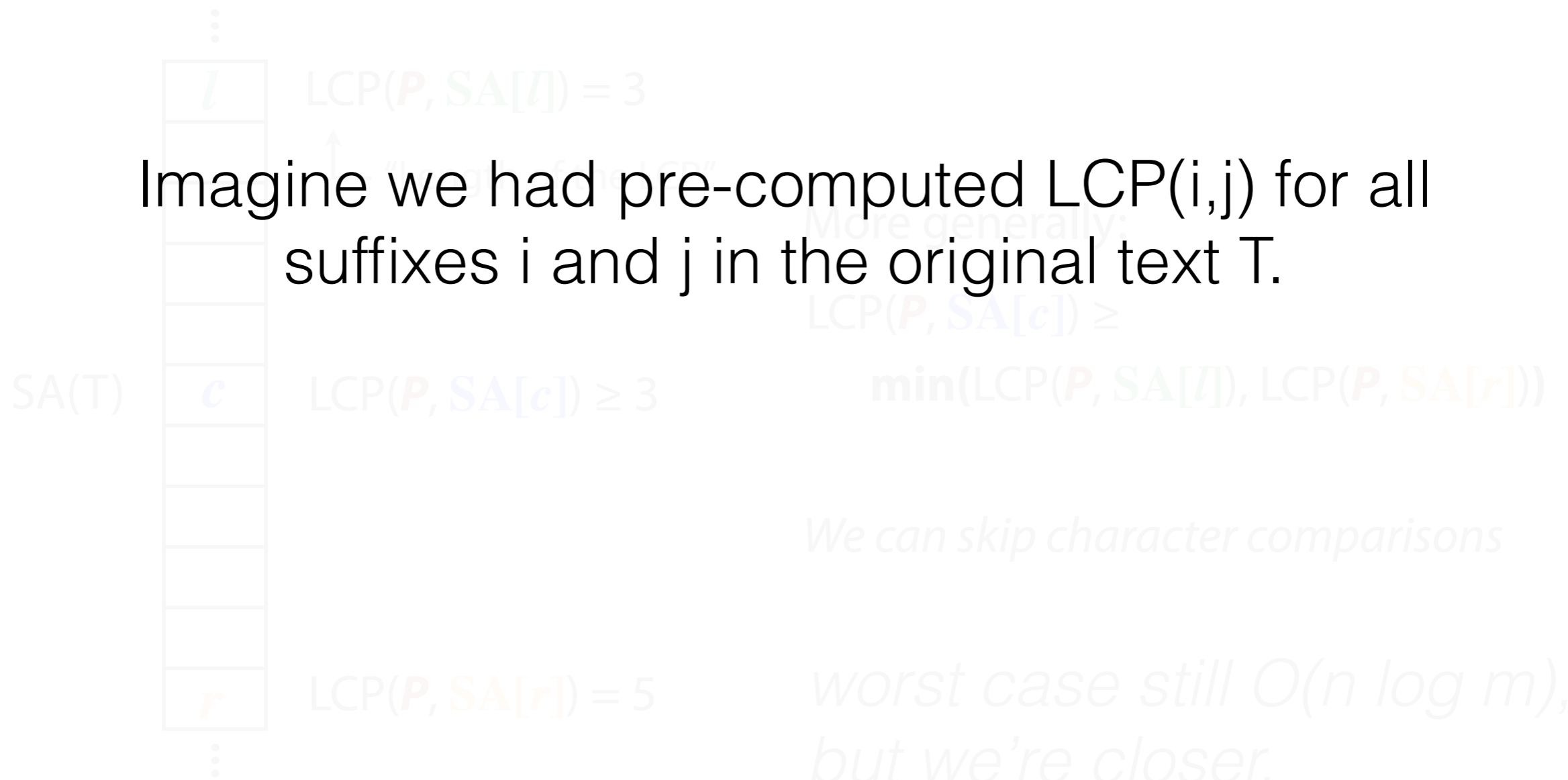
*worst case still $O(n \log m)$,
but we're closer.*

worst case example
 $S = ac_{M-2}b, P = c$

Suffix array: querying

Say we're comparing P to $SA[c]$ and we've already compared P to $SA[l]$ and $SA[r]$ in previous iterations.

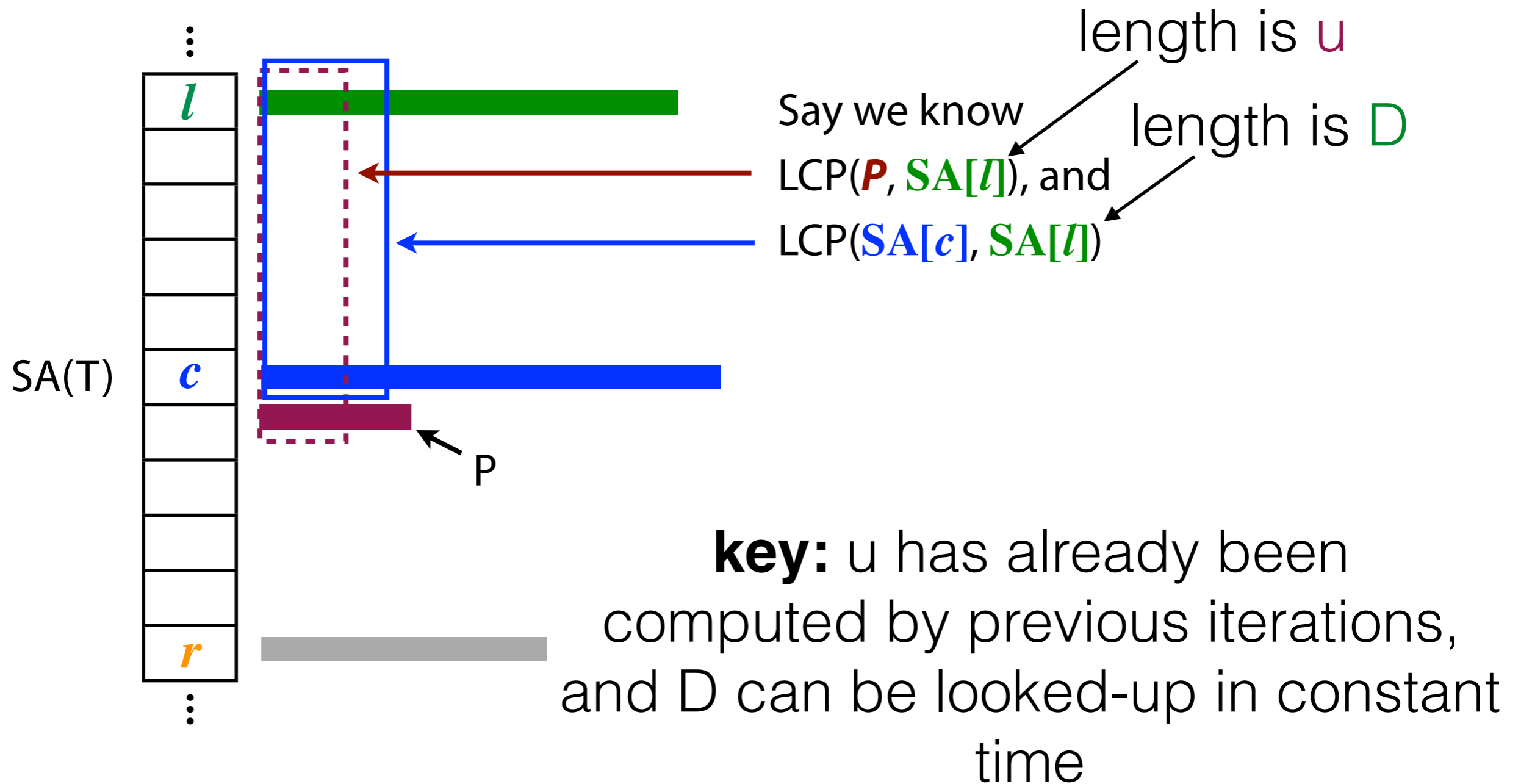
Imagine we had pre-computed $LCP(i,j)$ for all suffixes i and j in the original text T .



Suffix array: querying

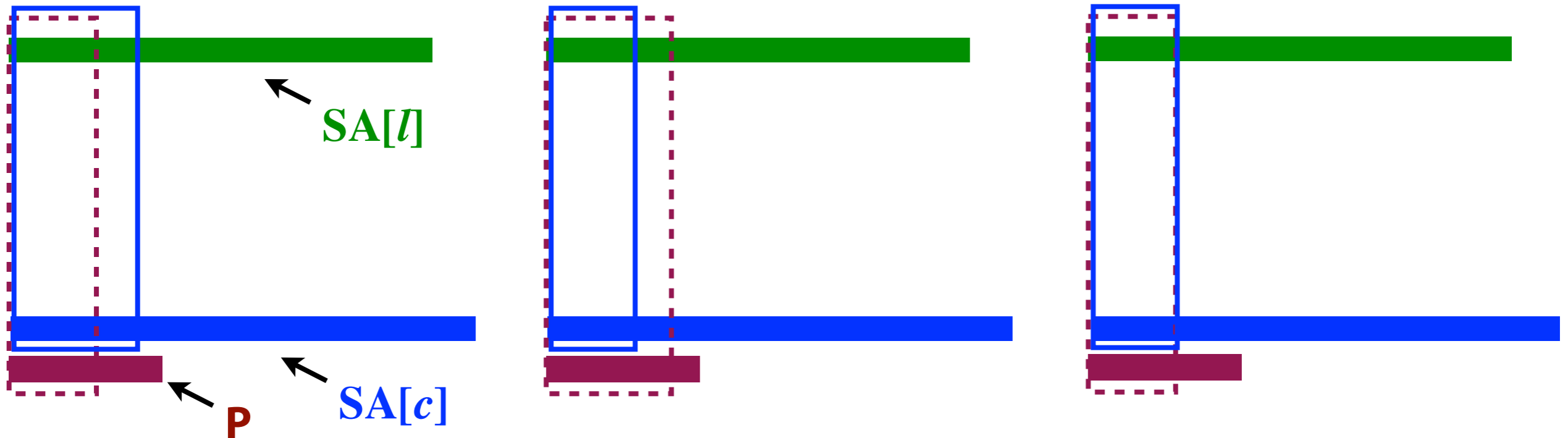
Take an iteration of binary search:

Assume, wlog, that $D = \text{LCP}(\text{SA}[l], \text{SA}[c]) \geq D' = \text{LCP}(\text{SA}[c], \text{SA}[r])$ otherwise there are symmetric cases.



Suffix array: querying

Three cases: or, if $D' = \text{LCP}(P, SA[r])$ is larger, 3 symmetric cases.



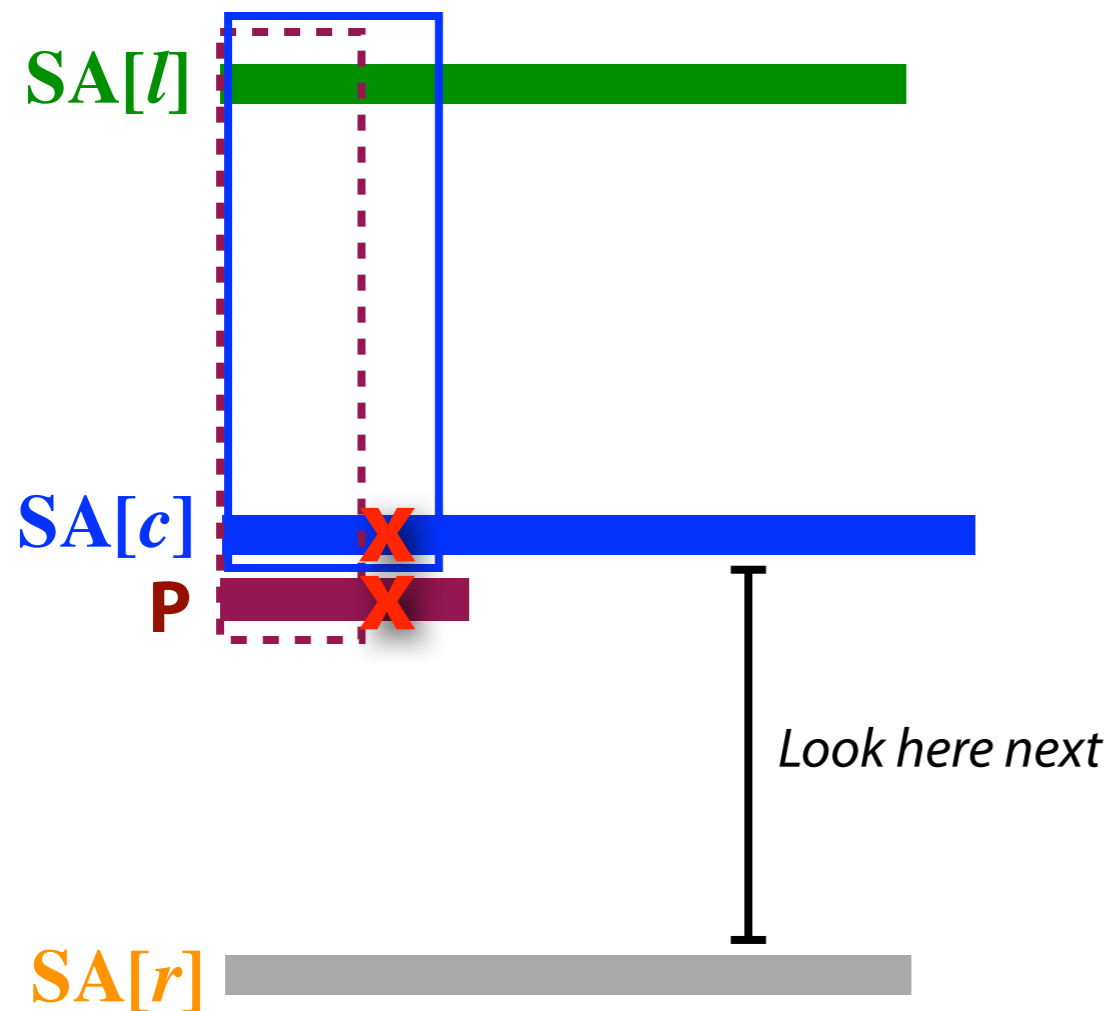
$$\text{LCP}(SA[c], SA[l]) > \text{LCP}(P, SA[l])$$

$$\text{LCP}(SA[c], SA[l]) < \text{LCP}(P, SA[l])$$

$$\text{LCP}(SA[c], SA[l]) = \text{LCP}(P, SA[l])$$

Suffix array: querying

Case 1: $\text{LCP}(\text{SA}[c], \text{SA}[l]) >$
 $\text{LCP}(P, \text{SA}[l])$



Next char of P after the $\text{LCP}(P, \text{SA}[l])$ must be *greater than* corresponding char of $\text{SA}[c]$

$$P > \text{SA}[c]$$

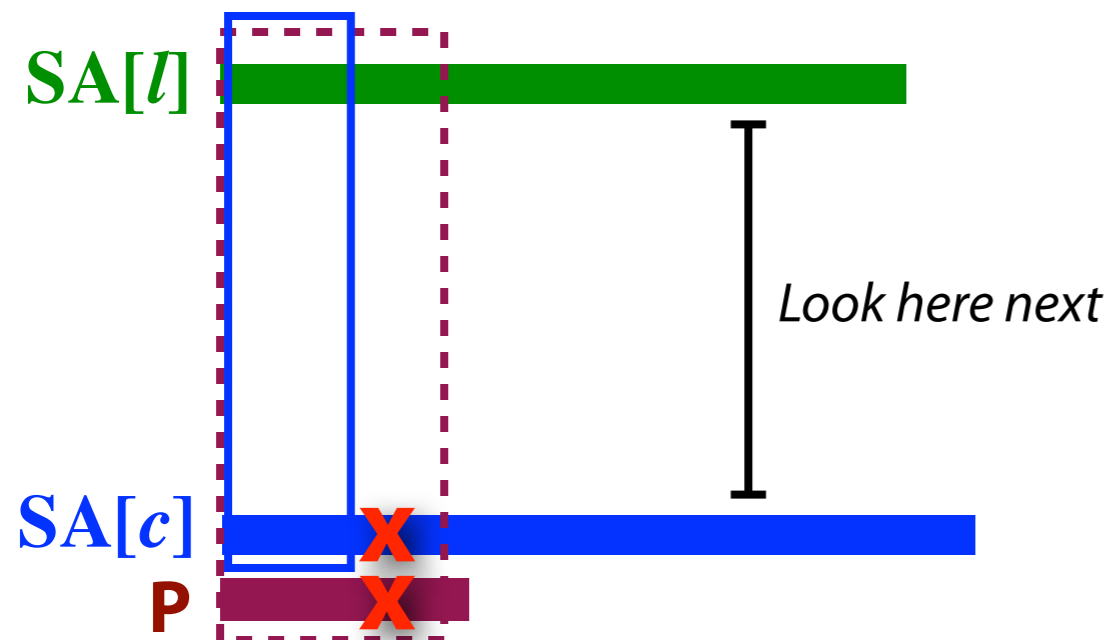
In this case, we compute $\text{LCP}(P[u:], \text{SA}[c][u:])$.
 c becomes our new l ,
and now we know that $\text{LCP}(P, \text{SA}[l])$, b/c we just computed it!

$$\text{LCP}(\text{SA}[c], \text{SA}[l]) >$$

$$\text{LCP}(P, \text{SA}[l])$$

Suffix array: querying

Case 2: $\text{LCP}(\text{SA}[c], \text{SA}[l]) < \text{LCP}(P, \text{SA}[l])$



Next char of $\text{SA}[c]$ after $\text{LCP}(\text{SA}[c], \text{SA}[l])$ must be *greater than* corresponding char of P

$P < \text{SA}[c]$

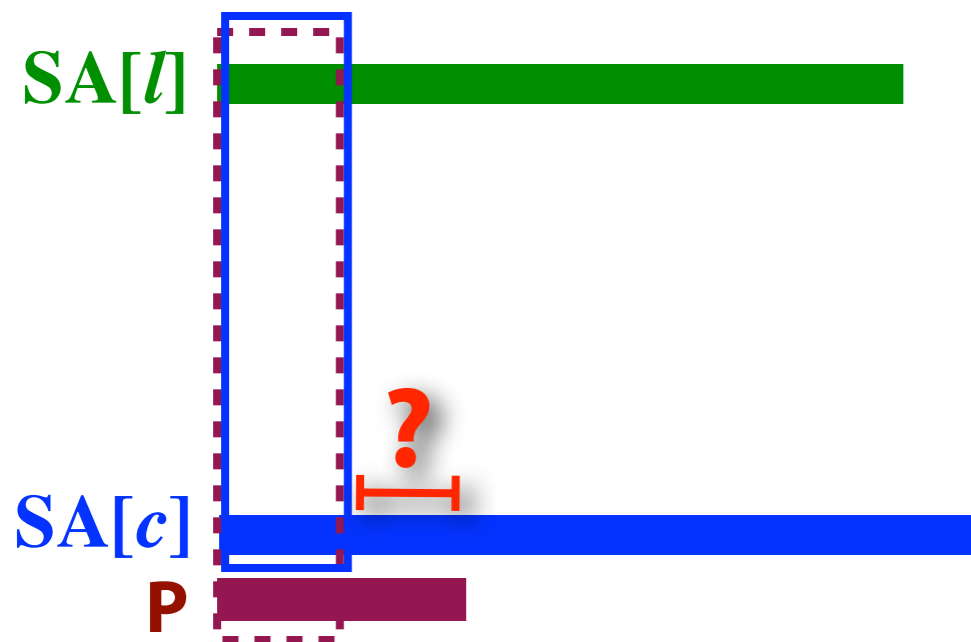
In this case, we compute $\text{LCP}(P[u:], \text{SA}[c][u:])$.
 c becomes our new r ,
 and now we know that $\text{LCP}(P, \text{SA}[r])$, b/c we just computed it!

$\text{SA}[r]$

$\text{LCP}(\text{SA}[c], \text{SA}[l]) < \text{LCP}(P, \text{SA}[l])$

Suffix array: querying

Case 3: $\text{LCP}(\text{SA}[c], \text{SA}[l]) =$
 $\text{LCP}(\mathbf{P}, \text{SA}[l])$



Must do further character comparisons between \mathbf{P} and $\text{SA}[c]$

Each such comparison either:

- (a) mismatches, leading to a bisection
- (b) matches, in which case $\text{LCP}(\mathbf{P}, \text{SA}[c])$ grows

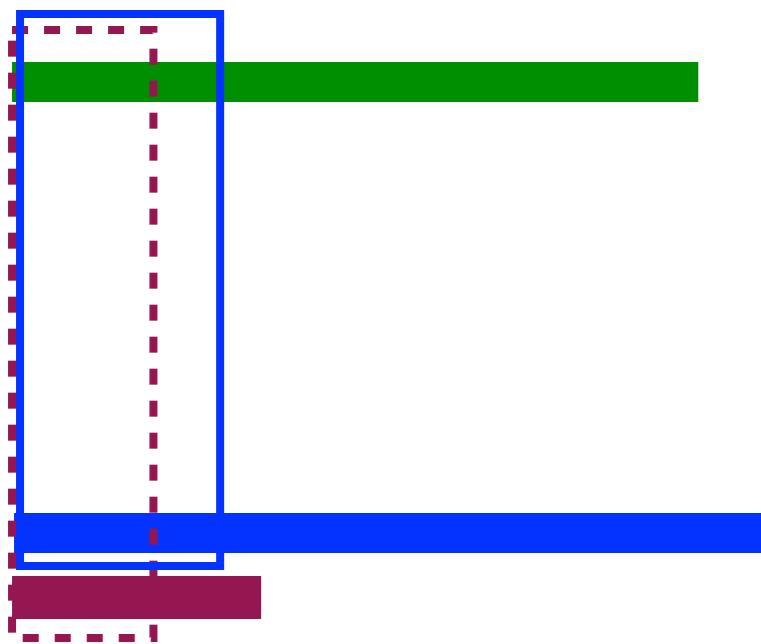
$\text{SA}[r]$ 

$$\text{LCP}(\text{SA}[c], \text{SA}[l]) =$$
$$\text{LCP}(\mathbf{P}, \text{SA}[l])$$

Suffix array: querying

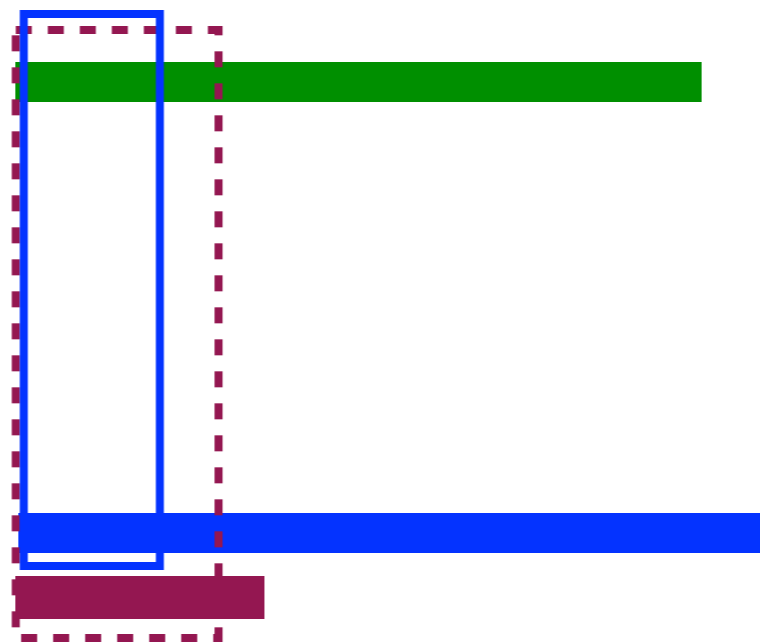
We improved binary search on suffix array from $O(n \log m)$ to $O(n + \log m)$ using information about Longest Common Prefixes (LCPs).

LCPs between P and suffixes of T computed during search, LCPs *among* suffixes of T computed *offline*



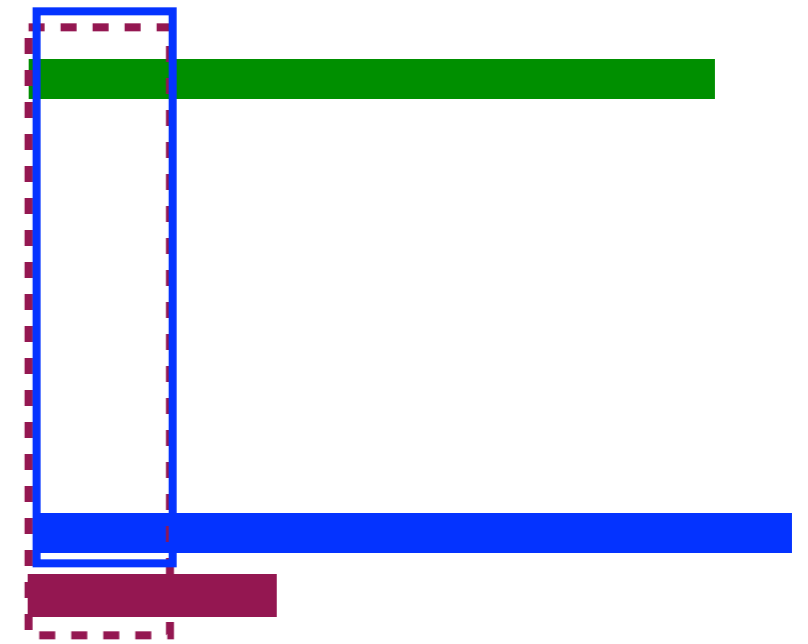
$$\text{LCP}(\text{SA}[c], \text{SA}[l]) > \\ \text{LCP}(P, \text{SA}[l])$$

Bisect right!



$$\text{LCP}(\text{SA}[c], \text{SA}[l]) < \\ \text{LCP}(P, \text{SA}[l])$$

Bisect left!



$$\text{LCP}(\text{SA}[c], \text{SA}[l]) = \\ \text{LCP}(P, \text{SA}[l])$$

Compare some
characters, then bisect!

Sketch of Running Time

Thm. Given the $LCP(X, Y)$ values, searching for a string P in a suffix array of length m now takes $O(|P| + \log m)$ time.

In case 1 & 2, we make $O(1)$ comparisons and bisect left or right — there are at most $O(\log m)$ bisections.

In case 3 we try to match characters starting at some offset between $SA[c]$ and P . If they match, those characters will never be compared again, so there are at most $O(|P|)$ such comparisons.

Mismatching characters may be compared more than once.

But there can be only 1 mismatch / bisection. There are $O(\log m)$ bisections, so there are at most $O(\log m)$ mismatches.

∴ Total # of comparisons = $O(|P| + \log m)$.



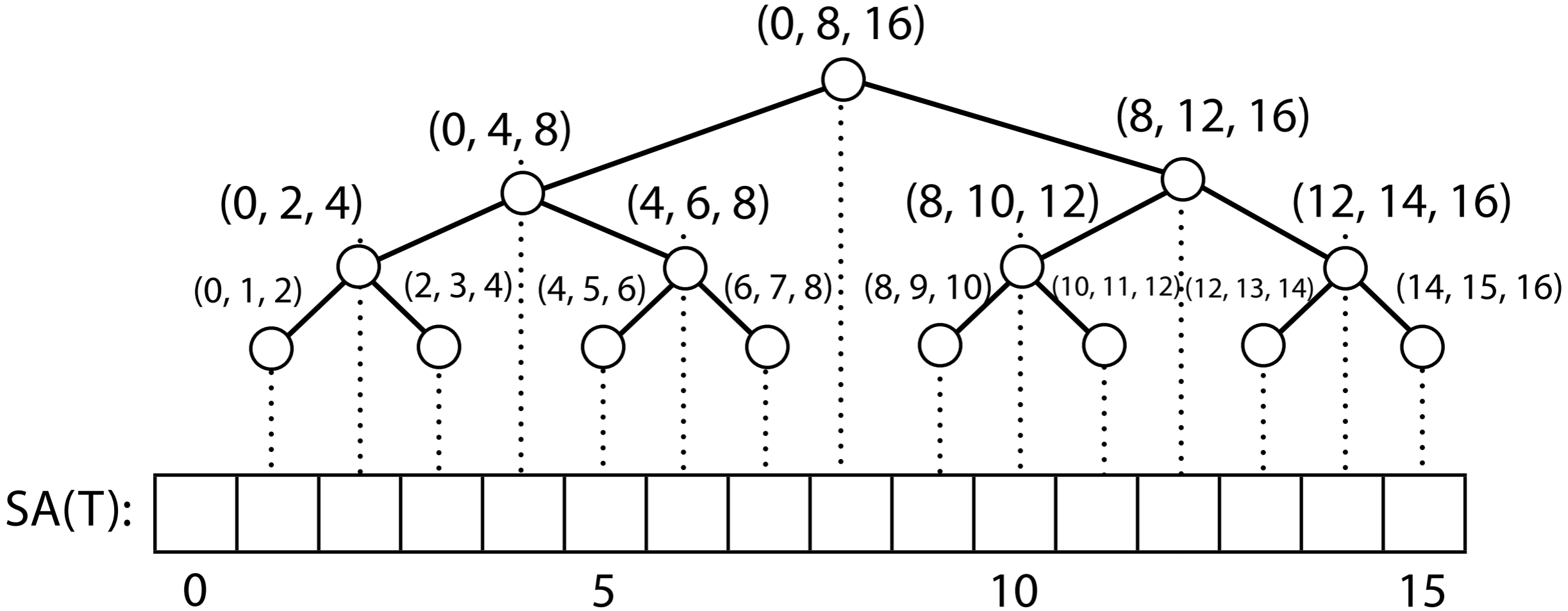
How to pre-compute LCP

- To perform this “efficient” search, we must be able to look up $\text{LCP}(\text{SA}[c], \text{SA}[l])$ and $\text{LCP}(\text{SA}[c], \text{SA}[r])$.
- How can we pre-compute this information *efficiently*?
 - Which LCP values do we need (*hint: not all of them*)?
 - Given LCP for left and right sub-interval of a search, how can we compute LCP for the containing interval?

Suffix array: LCPs

How to pre-calculate LCPs for every (l, c) and (c, r) pair in the search tree?

Triples are (l, c, r) triples

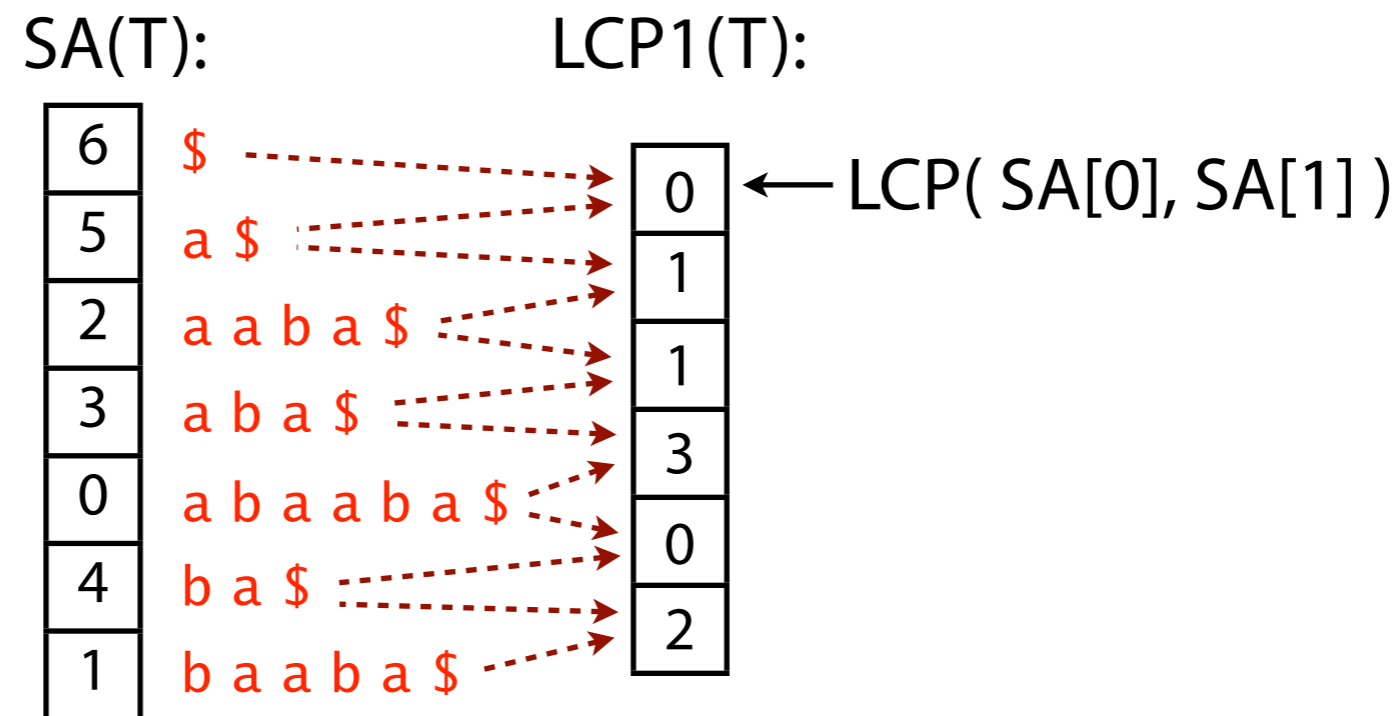


Example where $m = 16$ (incl. \$) # search tree nodes = $m - 1$

Suffix array: LCPs

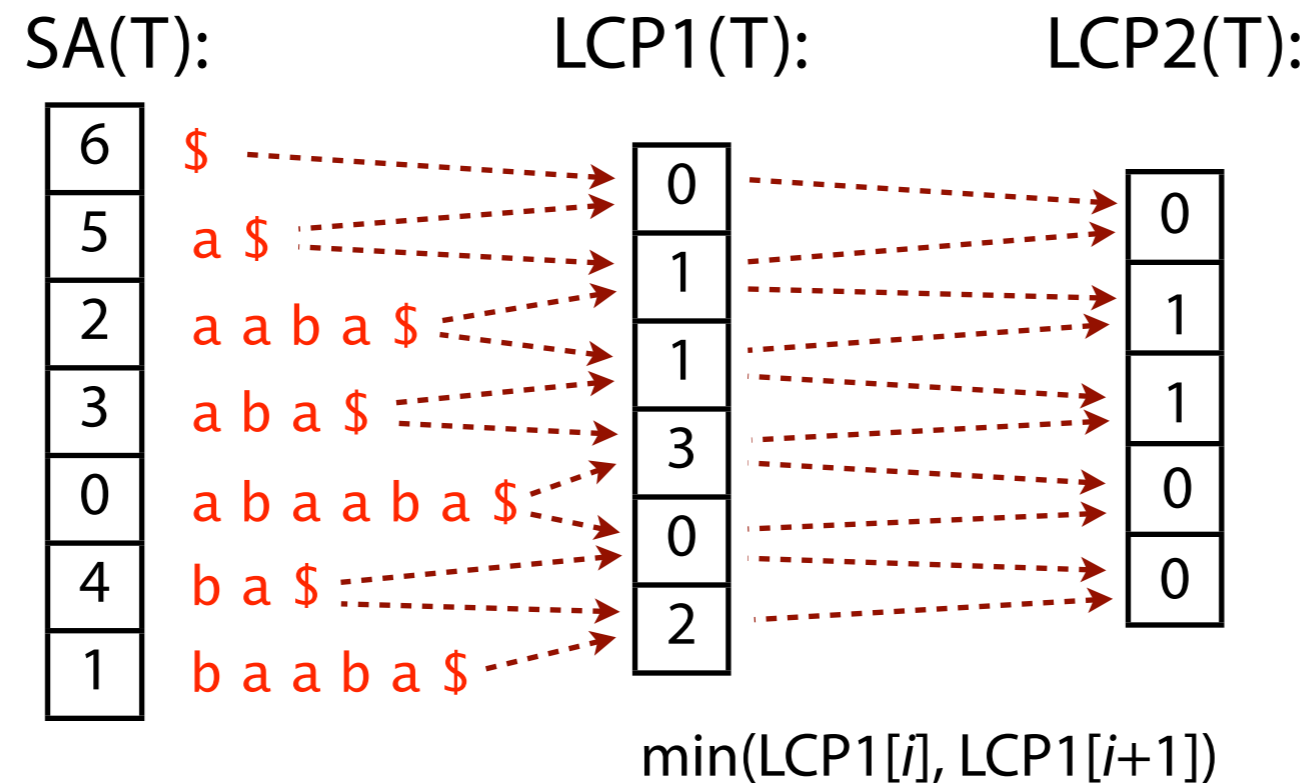
Suffix Array (SA) has m elements

Define LCP1 array with $m - 1$ elements such that $LCP[i] = LCP(SA[i], SA[i+1])$



Suffix array: LCPs

$$\text{LCP2}[i] = \text{LCP}(\text{SA}[i], \text{SA}[i+1], \text{SA}[i+2])$$

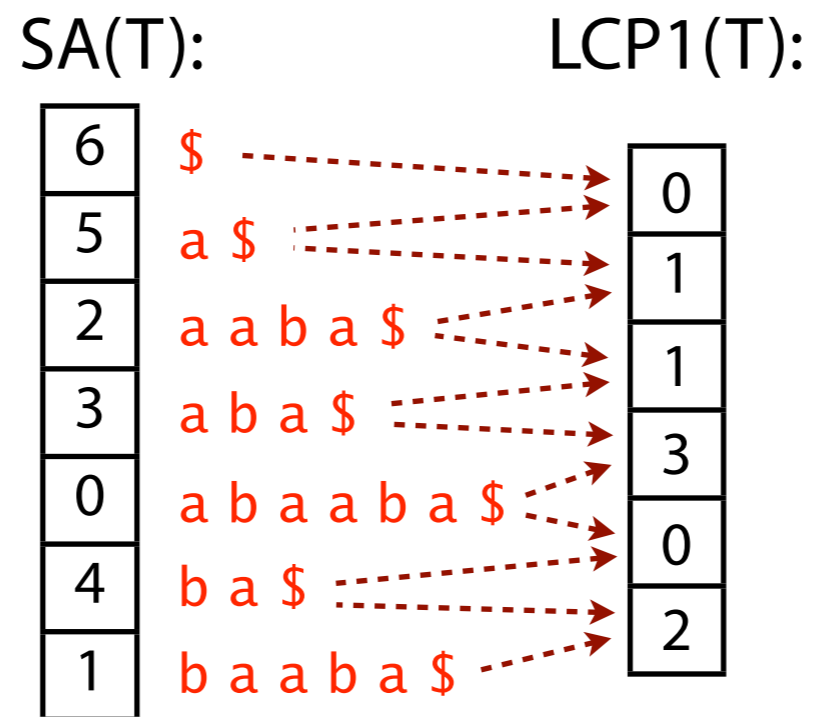


In fact, LCP of a range of consecutive suffixes in SA equals the minimum LCP1 among adjacent pairs in the range

LCP1 is a building block for other useful LCPs

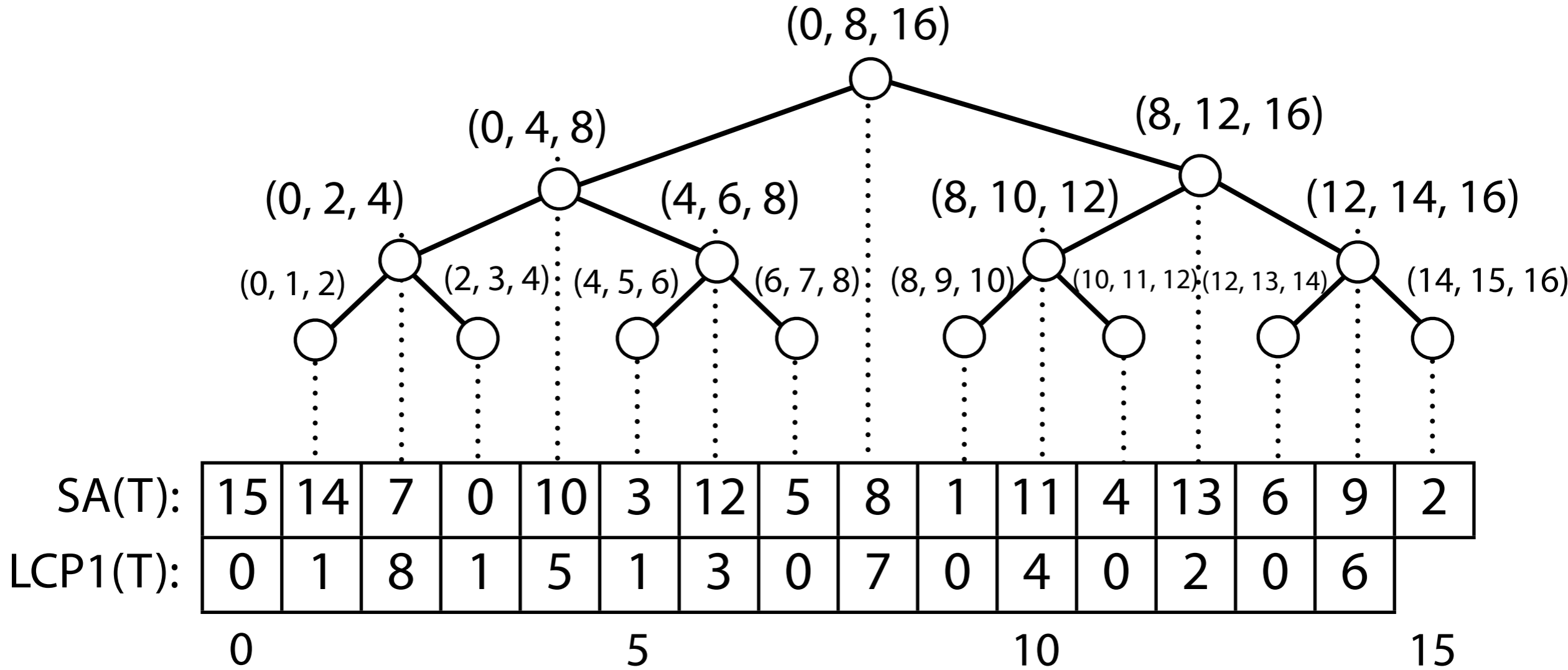
Suffix array: LCPs

Good time to calculate LCP1 it is *at the same time* as we *build* the suffix array, since putting the suffixes in order involves breaking ties after common prefixes



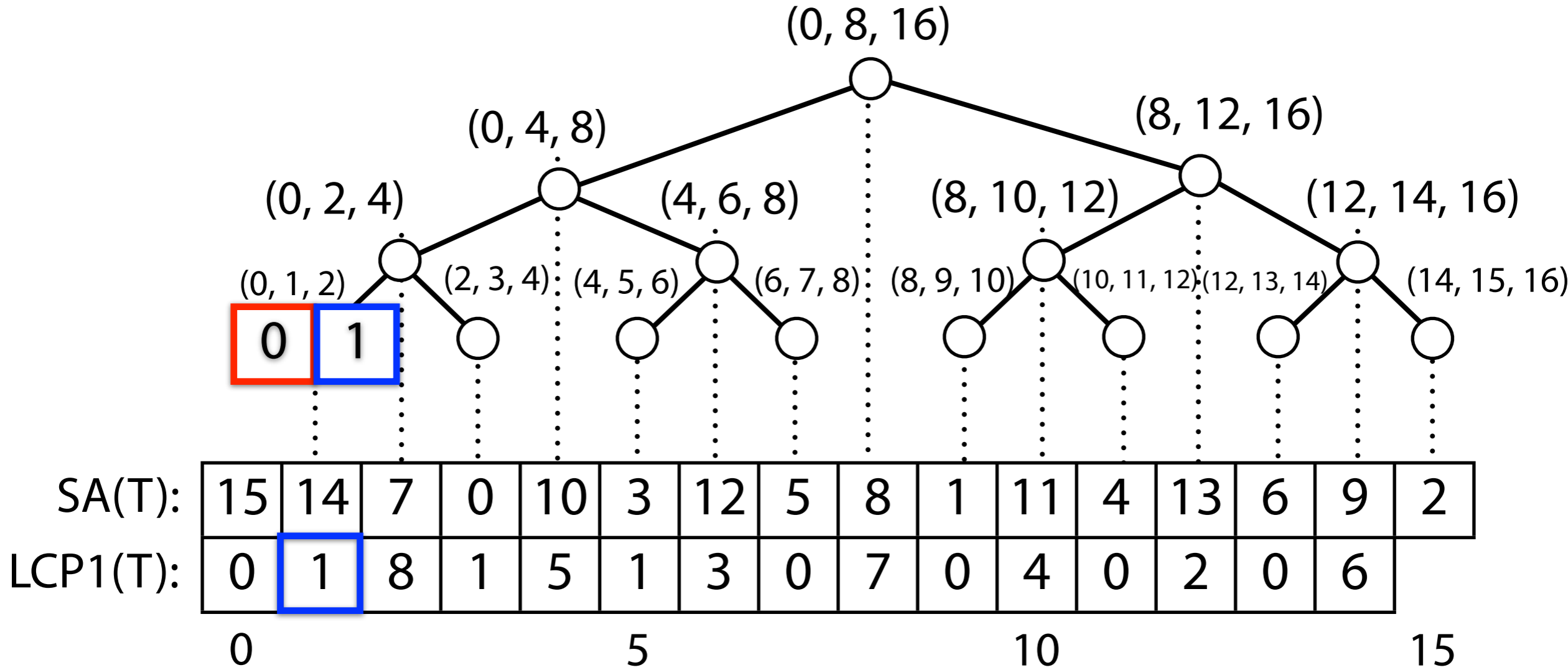
Suffix array: LCPs

T = abracadabracada



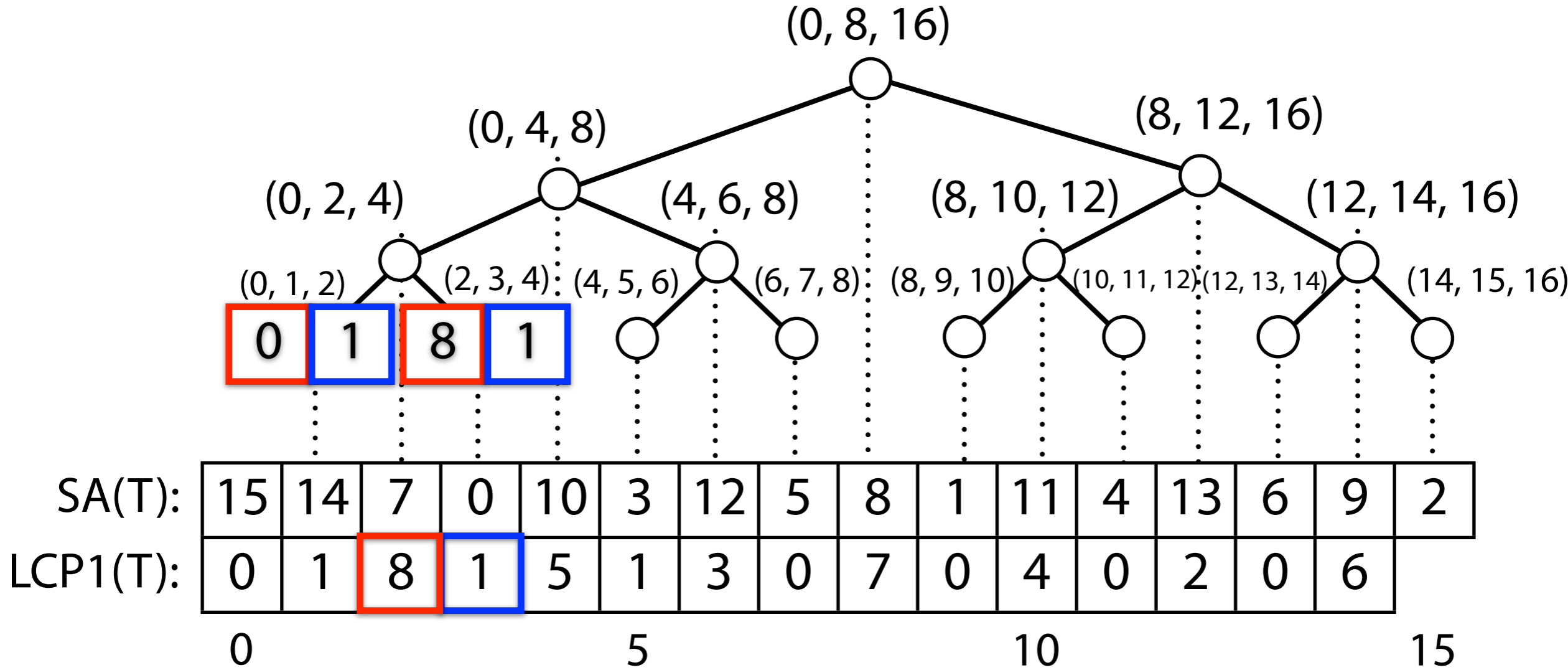
Suffix array: LCPs

T = abracadabracada\$



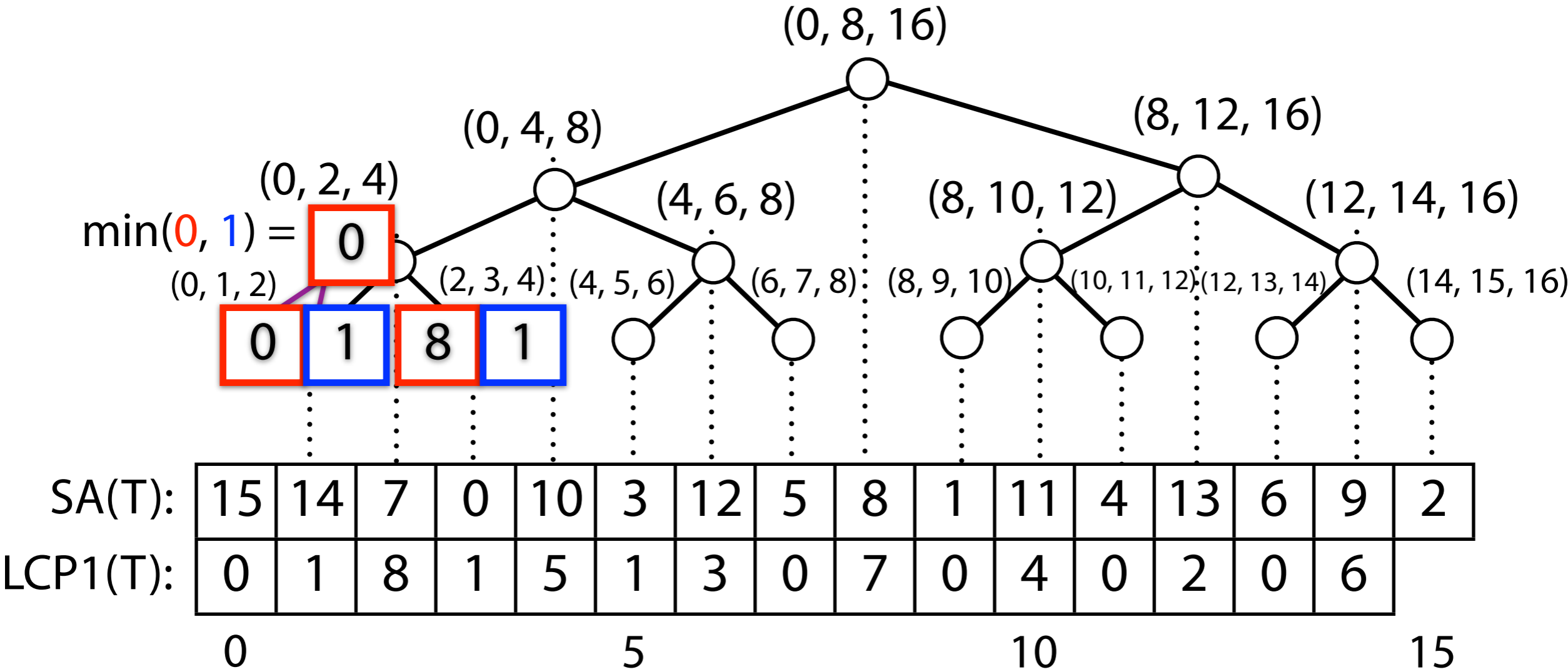
Suffix array: LCPs

T = abracadabracada\$



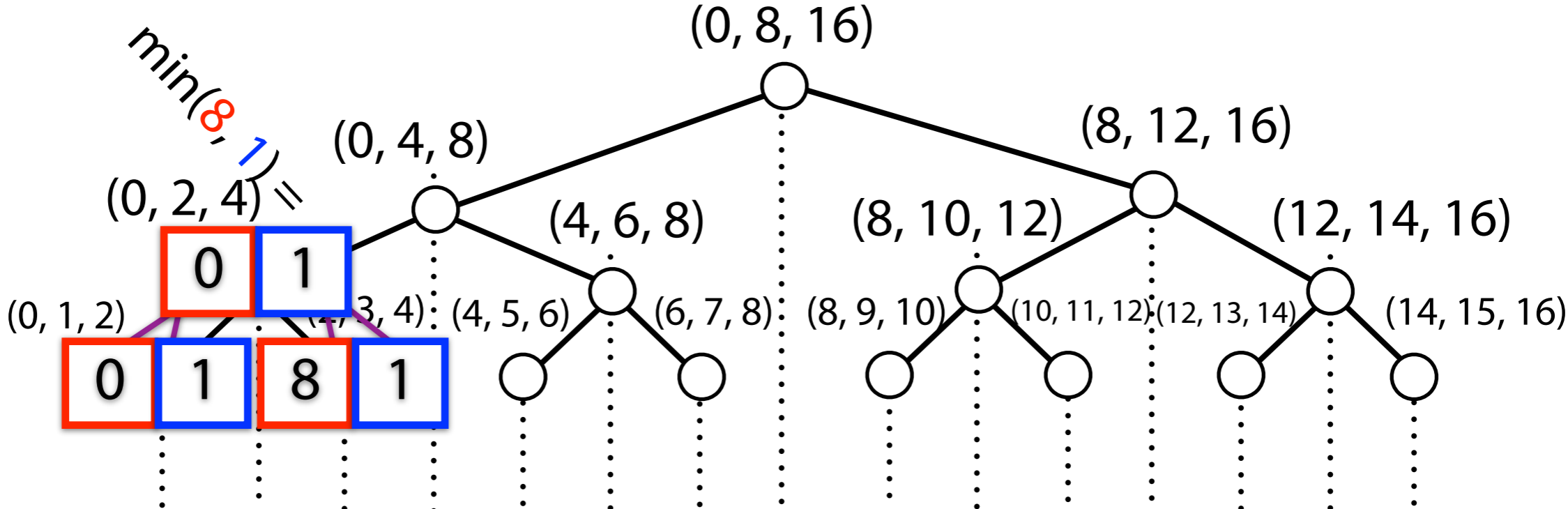
Suffix array: LCPs

T = abracadabracada\$



Suffix array: LCPs

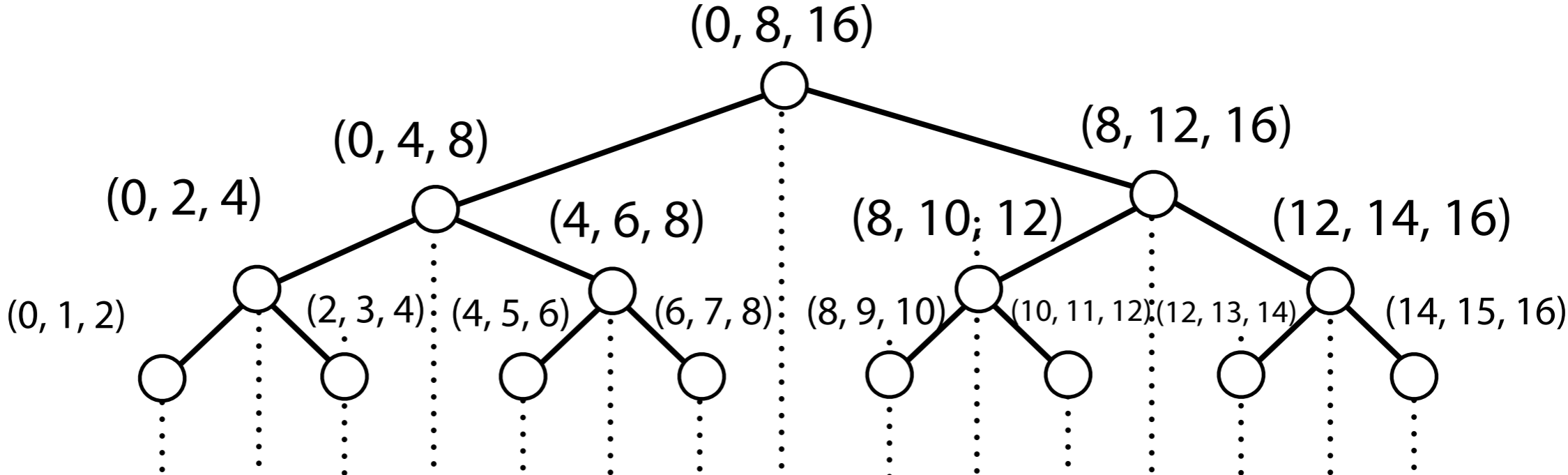
T = abracadabracada\$



SA(T):	15	14	7	0	10	3	12	5	8	1	11	4	13	6	9	2
LCP1(T):	0	1	8	1	5	1	3	0	7	0	4	0	2	0	6	
LCP_LC(T):	0	0	8													
LCP_CR(T):	1	1	1													
	0				5					10						15

Suffix array: LCPs

T = abracadabracada\$



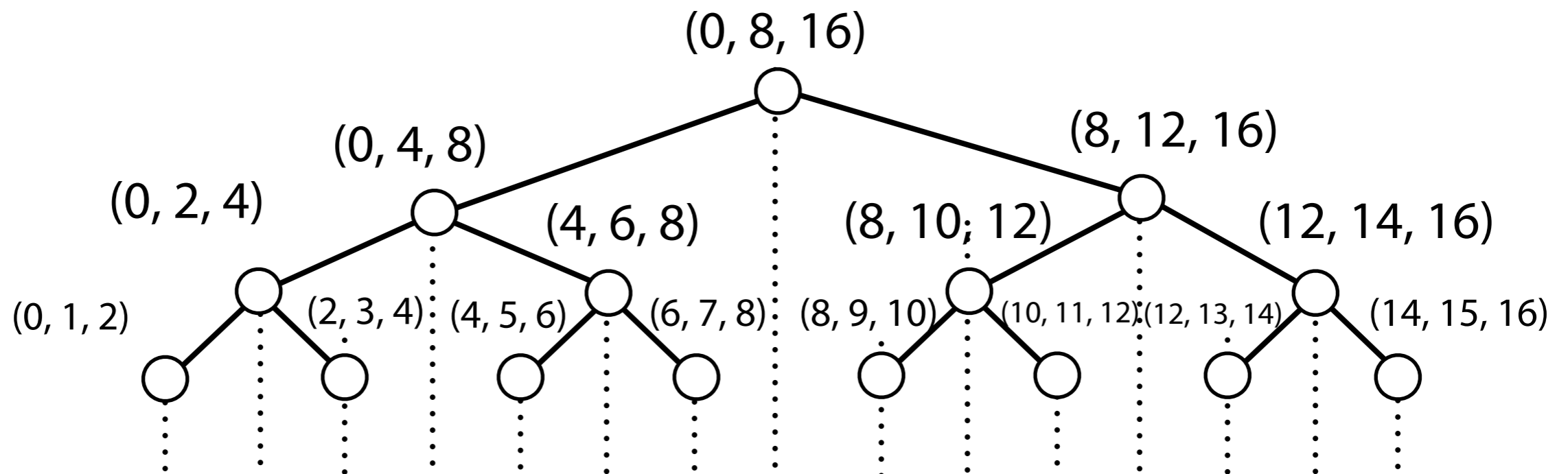
SA(T):	15	14	7	0	10	3	12	5	8	1	11	4	13	6	9	2
LCP1(T):	0	1	8	1	5	1	3	0	7	0	4	0	2	0	6	
LCP _{LC} (T):	0	0	8	0	5	1	3	0	7	0	4	0	2	0	6	
LCP _{CR} (T):	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	
	0				5				10						15	

NOTE: These arrays are “shifted” by 1 — the value in LCP_{LC} corresponding to (0, 1, 2) is at LCP_{LC}[0], not LCP_{LC}[1]. So, to look up LCP(SA[l], SA[c]) we look at LCP_{LC}[c-1]

Suffix array: LCPs

Can be done in:
 $O(m)$ time and space

T = abracadabracada\$



SA(T):	15	14	7	0	10	3	12	5	8	1	11	4	13	6	9	2
LCP1(T):	0	1	8	1	5	1	3	0	7	0	4	0	2	0	6	
LCP _{LC} (T):	0	0	8	0	5	1	3	0	7	0	4	0	2	0	6	
LCP _{CR} (T):	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	
	0				5					10					15	

NOTE: These arrays are “shifted” by 1 — the value in LCP_{LC} corresponding to (0, 1, 2) is at LCP_{LC}[0], not LCP_{LC}[1]. So, to look up LCP(SA[i], SA[c]) we look at LCP_{LC}[c-1]

Suffix array: querying review

We saw 3 ways to query (binary search) the suffix array:

1. Typical binary search. Ignores LCPs. $O(n \log m)$.
2. Binary search with some skipping using LCPs between P and T 's suffixes. Still $O(n \log m)$, but it can be argued it's near $O(n + \log m)$ in practice. Gusfield: "Simple Accelerant"
3. Binary search with skipping using all LCPs, including LCPs among T 's suffixes. $O(n + \log m)$. Gusfield: "Super Accelerant"

How much space do they require?

1. $\sim m$ integers (SA)
2. $\sim m$ integers (SA)
3. $\sim 3m$ integers (SA, LCP_LC, LCP_CR)

Suffix array: performance comparison

	Super accelerant	Simple accelerant	No accelerant
python -O	68.78 s	69.80 s	102.71 s
pypy -O	5.37 s	5.21 s	8.74 s
# character comparisons	99.5 M	117 M	235 M

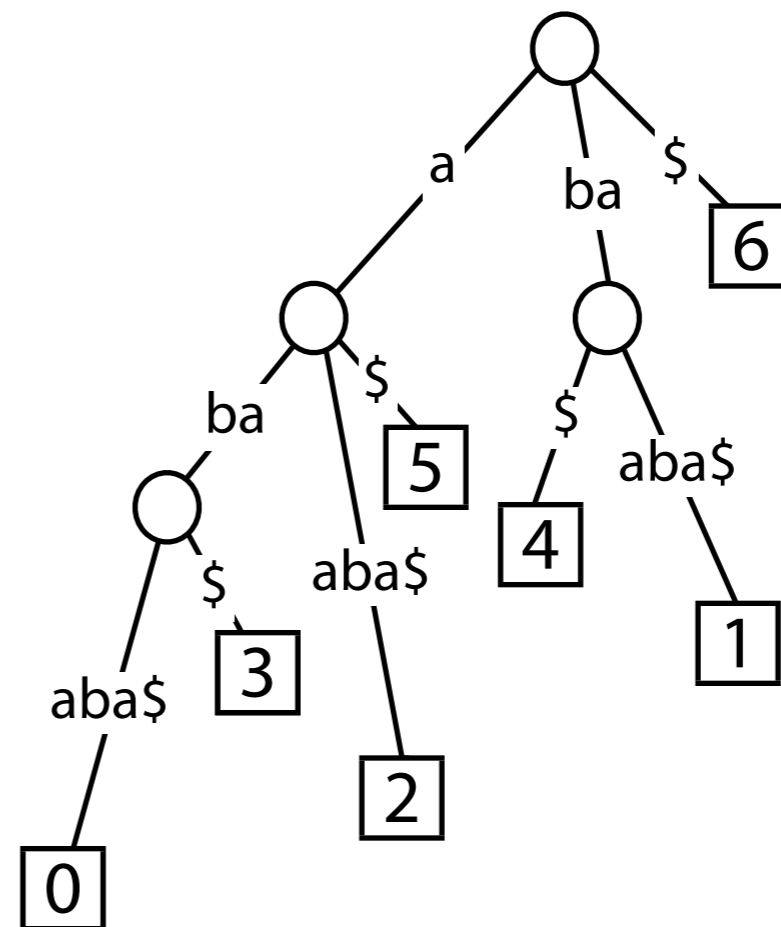
Matching 500K 100-nt substrings to the ~ 5 million nt-long *E. coli* genome. Substrings drawn randomly from the genome.

Index building time not included

Suffix array: building

Given T , how to we efficiently build T 's suffix array?

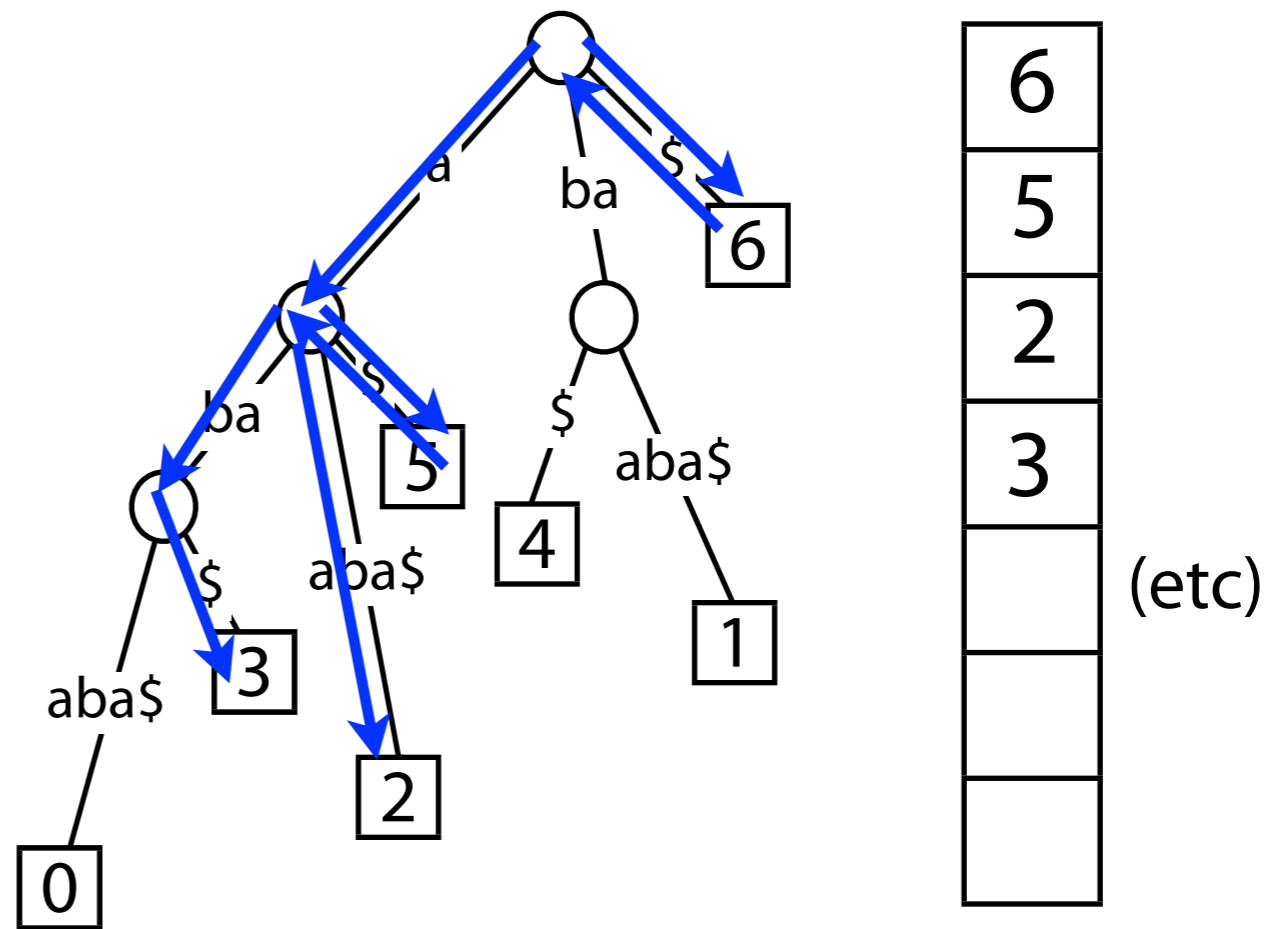
6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$



Suffix array: building SA

Idea: Build suffix tree, do a lexicographic depth-first traversal reporting leaf offsets as we go

Traverse $O(m)$ nodes and emit m integers, so $O(m)$ time assuming edges are already ordered



Suffix array: building

Suffix trees are big. Given T , how do we efficiently build T 's suffix array *without* first building a suffix tree?

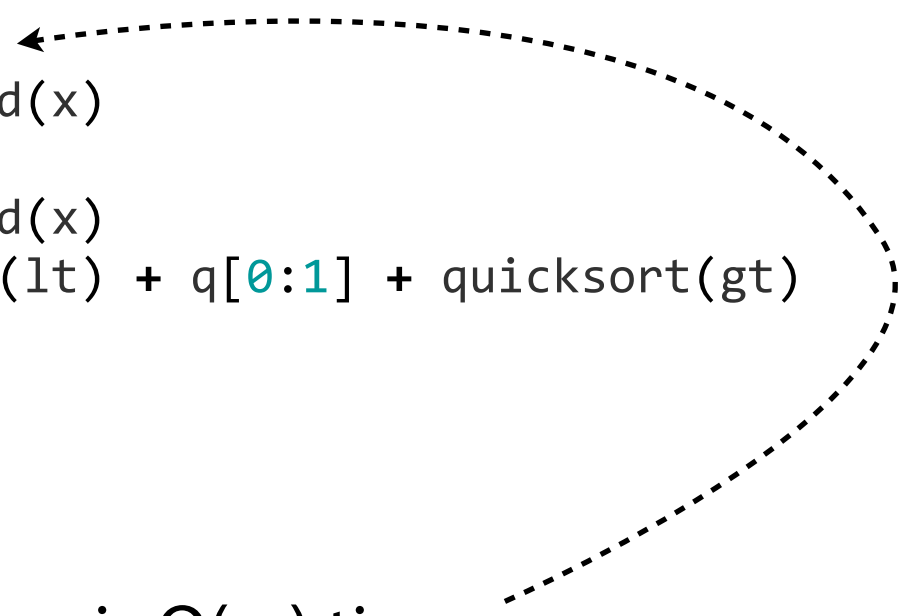
6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

Suffix array: sorting suffixes

One idea: Use your favorite sort, e.g., quicksort

0	a b a a b a \$
1	b a a b a \$
2	a a b a \$
3	a b a \$
4	b a \$
5	a \$
6	\$

```
def quicksort(q):  
    lt, gt = [], []  
    if len(q) <= 1:  
        return q  
    for x in q[1:]:  
        if x < q[0]:  
            lt.append(x)  
        else:  
            gt.append(x)  
    return quicksort(lt) + q[0:1] + quicksort(gt)
```



Expected time: $O(m^2 \log m)$

Not $O(m \log m)$ because a suffix comparison is $O(m)$ time

Suffix array: sorting suffixes

One idea: Use a sort algorithm that's aware that the items being sorted are strings, e.g. "multikey quicksort"

0	a b a a b a \$
1	b a a b a \$
2	a a b a \$
3	a b a \$
4	b a \$
5	a \$
6	\$

Essentially $O(m^2)$ time

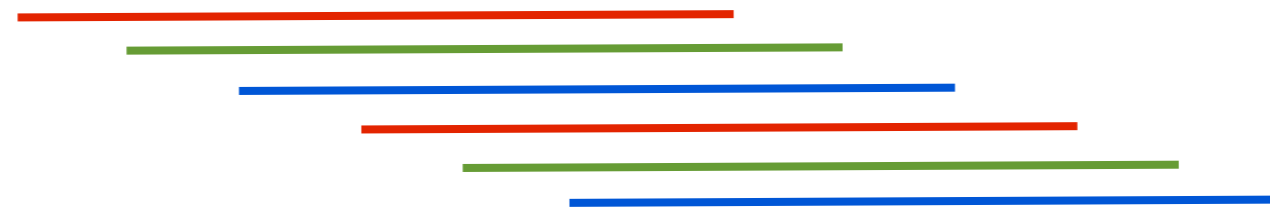
Bentley, Jon L., and Robert Sedgwick. "Fast algorithms for sorting and searching strings." *Proceedings of the eighth annual ACM-SIAM symposium on Discrete algorithms*. Society for Industrial and Applied Mathematics, 1997

The Skew Algorithm (aka DC3)

Kärkkäinen & Sanders, 2003

- **Main idea: Divide suffixes into 3 groups:**
 - Those starting at positions $i=0,3,6,9,\dots$ ($i \bmod 3 = 0$)
 - Those starting at positions $1,4,7,10,\dots$ ($i \bmod 3 = 1$)
 - Those starting at positions $2,5,8,11,\dots$ ($i \bmod 3 = 2$)
- For simplicity, assume text length is a multiple of 3 after padding with a special character.

$T[0, n) =$ 0 1 2 3 4 5 6 7 8 9 10 11
y a b b a d a b b a d o



$SA = (12, 1, 6, 4, 9, 3, 8, 2, 7, 5, 10, 11, 0)$

Basic Outline:

- Recursively handle suffixes from the $i \bmod 3 = 1$ and $i \bmod 3 = 2$ groups.
- Merge the $i \bmod 3 = 0$ group at the end.

Step 0 — Constructing a sample

These are called the “sample suffixes”

Step 0: Construct a sample. For $k = 0, 1, 2$, define

$$B_k = \{i \in [0, n] \mid i \bmod 3 = k\}.$$

Let $C = B_1 \cup B_2$ be the set of *sample positions* and S_C the set of *sample suffixes*.

Example. $B_1 = \{1, 4, 7, 10\}$, $B_2 = \{2, 5, 8, 11\}$, i.e., $C = \{1, 4, 7, 10, 2, 5, 8, 11\}$.

Step 1 — Sorting the sample

Step 1: Sort sample suffixes. For $k = 1, 2$, construct the strings

$$R_k = [t_k t_{k+1} t_{k+2}][t_{k+3} t_{k+4} t_{k+5}] \cdots [t_{\max B_k} t_{\max B_k + 1} t_{\max B_k + 2}]$$

whose characters are triples $[t_i t_{i+1} t_{i+2}]$. Note that the last character of R_k is always unique because $t_{\max B_k + 2} = 0$. Let $R = R_1 \odot R_2$ be the concatenation of R_1 and R_2 . Then the (nonempty) suffixes of R correspond to the set S_C of sample suffixes: $[t_i t_{i+1} t_{i+2}][t_{i+3} t_{i+4} t_{i+5}] \cdots$ corresponds to S_i . The correspondence is order preserving, i.e., by sorting the suffixes of R we get the order of the sample suffixes S_C .

Example. $R = [\text{abb}][\text{ada}][\text{bba}][\text{do0}][\text{bba}][\text{dab}][\text{bad}][\text{o00}]$.

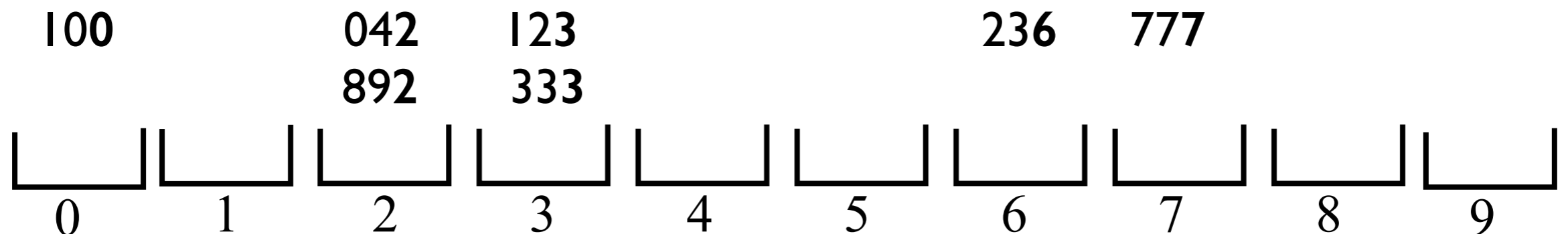
Step 1 — Sorting the sample

To sort the suffixes of R , first radix sort the characters of R and rename them with their ranks to obtain the string R' . If all characters are different, the order of characters gives directly the order of suffixes. Otherwise, sort the suffixes of R' using Algorithm DC3.

Example. $R' = (1, 2, 4, 6, 4, 5, 3, 7)$ and $SA_{R'} = (8, 0, 1, 6, 4, 2, 5, 3, 7)$.

Interlude: Radix Sort

- $O(n)$ -time sort for n items when items can be divided into constant # of digits.
- Put into buckets based on least-significant digit, flatten, repeat with next-most significant digit, etc.
- Example items: 100 123 042 333 777 892 236



- # of passes = # of digits
- Each pass goes through the numbers once.

Step 1.5 — Sorting the sample

Example. $R = [\text{abb}][\text{ada}][\text{bba}][\text{do0}][\text{bba}][\text{dab}][\text{bad}][\text{o00}]$.

Once the sample suffixes are sorted, assign a rank to each suffix. For $i \in C$, let $\text{rank}(S_i)$ denote the rank of S_i in the sample set S_C . Additionally, define $\text{rank}(S_{n+1}) = \text{rank}(S_{n+2}) = 0$. For $i \in B_0$, $\text{rank}(S_i)$ is undefined.

Example. $\text{rank}(S_i)$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	\perp	1	4	\perp	2	6	\perp	5	3	\perp	7	8	\perp	0	0

Step 1.5 — Sorting the sample

Once the sample suffixes are sorted, assign a rank to each suffix. For $i \in C$, let $rank(S_i)$ denote the rank of S_i in the sample set S_C . Additionally, define $rank(S_{n+1}) = rank(S_{n+2}) = 0$. For $i \in B_0$, $rank(S_i)$ is undefined.

Example. $rank(S_i)$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	\perp	1	4	\perp	2	6	\perp	5	3	\perp	7	8	\perp	0	0

$T[0, n) =$

i	0	1	2	3	4	5	6	7	8	9	10	11
	y	a	b	b	a	d	a	b	b	a	d	o

Example. $R = [abb][ada][bba][do0][bba][dab][bad][o00]$.

Example. $rank(S_i)$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	\perp	1	4	\perp	2	6	\perp	5	3	\perp	7	8	\perp	0	0

Step 1.5 — Sorting the sample

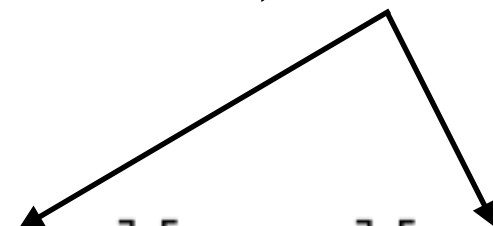
Once the sample suffixes are sorted, assign a rank to each suffix. For $i \in C$, let $rank(S_i)$ denote the rank of S_i in the sample set S_C . Additionally, define $rank(S_{n+1}) = rank(S_{n+2}) = 0$. For $i \in B_0$, $rank(S_i)$ is undefined.

Example.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$rank(S_i)$	\perp	1	4	\perp	2	6	\perp	5	3	\perp	7	8	\perp	0	0

Note: After only 1 level of recursion, these suffixes would be “tied”

Example. $R = [abb][ada][bba][do0][bba][dab][bad][o00]$.



The resolved ranks here represent what we'd get after a second level of recursion.

Step 1.5 — Sorting the sample

$T[0, n) =$

0	1	2	3	4	5	6	7	8	9	10	11
y	a	b	b	a	d	a	b	b	a	d	o

Example. $R = [abb][ada][bba][do0][bba][dab][bad][o00]$.

$R_2 = [247][463][474][638]$

These suffixes were tied at the previous level, but here, we can resolve them. The *lexical renaming* allows us to compare longer and longer suffixes of the text.

Step 2 — Sorting the non-sample suffixes

Step 2: Sort nonsample suffixes. Represent each nonsample suffix $S_i \in S_{B_0}$ with the pair $(t_i, \text{rank}(S_{i+1}))$. Note that $\text{rank}(S_{i+1})$ is always defined for $i \in B_0$. Clearly we have, for all $i, j \in B_0$,

$$S_i \leq S_j \iff (t_i, \text{rank}(S_{i+1})) \leq (t_j, \text{rank}(S_{j+1})).$$

The pairs $(t_i, \text{rank}(S_{i+1}))$ are then radix sorted.

Example. $S_{12} < S_6 < S_9 < S_3 < S_0$ because $(0, 0) < (\mathbf{a}, 5) < (\mathbf{a}, 7) < (\mathbf{b}, 2) < (\mathbf{y}, 1)$.

Step 2 — Sorting the non-sample suffixes

Step 3: Merge. The two sorted sets of suffixes are merged using a standard comparison-based merging. To compare suffix $S_i \in S_C$ with $S_j \in S_{B_0}$, we distinguish two cases:

$$i \in B_1 : \quad S_i \leq S_j \iff (t_i, \text{rank}(S_{i+1})) \leq (t_j, \text{rank}(S_{j+1}))$$

$$i \in B_2 : \quad S_i \leq S_j \iff (t_i, t_{i+1}, \text{rank}(S_{i+2})) \leq (t_j, t_{j+1}, \text{rank}(S_{j+2}))$$

Note that the ranks are defined in all cases.

Example. $S_1 < S_6$ because $(\mathbf{a}, 4) < (\mathbf{a}, 5)$ and $S_3 < S_8$ because $(\mathbf{b}, \mathbf{a}, 6) < (\mathbf{b}, \mathbf{a}, 7)$.

Running Time

$$T(n) = O(n) + T(2n/3)$$

time to sort and
merge

array in recursive calls
is 2/3rds the size of
starting array

Solves to $T(n) = O(n)$:

- Expand big-O notation: $T(n) \leq cn + T(2n/3)$ for some c .
- Guess: $T(n) \leq 3cn$
- Induction step: assume that is true for all $i < n$.
- $T(n) \leq cn + 3c(2n/3) = cn + 2cn = 3cn \quad \square$

Handling the 1 and 2 groups

These are called the “sample suffixes”

s = mississippi\$\$

iss	iss	ipp	i\$\$	ssi	ssi	ppi
-----	-----	-----	-------	-----	-----	-----

triples for groups
1 and 2 groups

t = C C B A E E D

Sort the triples using a **radix** sort, then **lexically name** them (i.e. assign each triple a token according to its lexicographical rank).

AEED	4
BAEED	3
CBAEED	2
CCBAEED	1
D	7
ED	6
EED	5

recursively compute the suffix array for tokenized string

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Every suffix of t corresponds to a suffix of s (maybe with some cruft at the end of it).

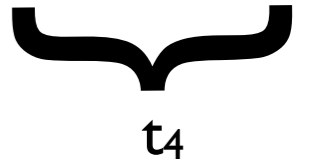
Relationship Between t and s

s = mississippi\$\$

iss	iss	ipp	i\$\$	ssi	ssi	ppi
-----	-----	-----	-------	-----	-----	-----

C C B A E E D

t = CCBAEED



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Key Point #1: The lexicographical order of the suffixes of t is the same as the order of the group 1 & 2 suffixes of s.

Why?

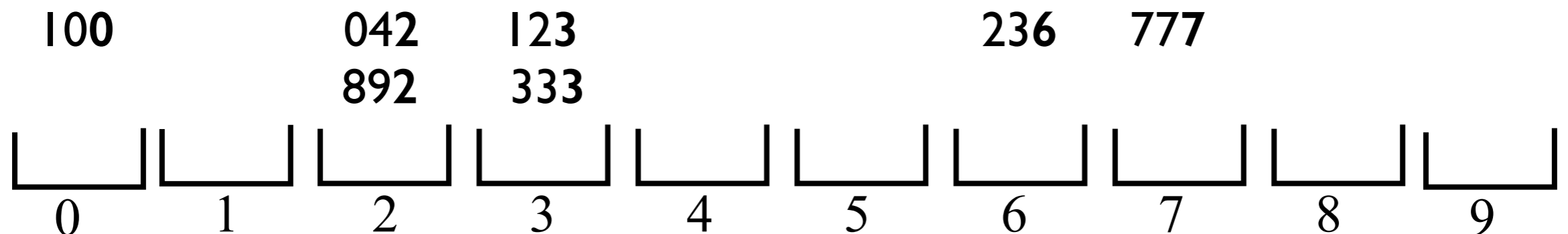
Every suffix of t corresponds to some suffix of s (perhaps with some extra letters at the end of it --- in this case EED)

Because the tokens are sorted in the same order as the triples, the sort order of the suffix of t matches that of s.

So: The recursive computational of the suffix array for t gives you the ordering of the group 1 and group 2 suffixes.

Radix Sort

- $O(n)$ -time sort for n items when items can be divided into constant # of digits.
- Put into buckets based on least-significant digit, flatten, repeat with next-most significant digit, etc.
- Example items: 100 123 042 333 777 892 236



- # of passes = # of digits
- Each pass goes through the numbers once.

Handling 0 Suffixes

- First: sort the group 0 suffixes, using the representation $(s[i], S_{i+1})$
- Since the S_{i+1} suffixes are already in the array sorted, we can just *stably* sort them with respect to $s[i]$, using radix sort.

1,2-array:

i	p	p	i	s	s	i	s	s	i	\$	\$	p	p	i	s	s	i	s	s	i
---	---	---	---	---	---	---	---	---	---	----	----	---	---	---	---	---	---	---	---	---

0-array:

m	i	s	p	i	\$	s	i	p	s	i	s
---	---	---	---	---	----	---	---	---	---	---	---

- We have to merge the group 0 suffixes into the suffix array for group 1 and 2.
- Given suffix S_i and S_j , need to decide which should come first.
 - If S_i and S_j are both either group 1 or group 2, then the recursively computed suffix array gives the order.
 - If one of i or j is $0 \pmod{3}$, see next slide.

Comparing 0 suffix S_j with 1 or 2 suffix S_i

Represent S_i and S_j using subsequent suffixes:

$i \pmod{3} = 1$:

$$(s[i], S_{i+1}) \stackrel{?}{<} (s[j], S_{j+1})$$

\uparrow \uparrow
 $\equiv 2 \pmod{3}$ $\equiv 1 \pmod{3}$

$i \pmod{3} = 2$:

$$(s[i], s[i+1], S_{i+2}) \stackrel{?}{<} (s[j], s[j+1], S_{j+2})$$

\uparrow \uparrow
 $\equiv 1 \pmod{3}$ $\equiv 2 \pmod{3}$

\Rightarrow the suffixes can be compared quickly because the relative order of S_{i+1}, S_{j+1} or S_{i+2}, S_{j+2} is known from the 1,2-array we already computed.

Running Time

$$T(n) = O(n) + T(2n/3)$$

time to sort and
merge

array in recursive calls
is 2/3rds the size of
starting array

Solves to $T(n) = O(n)$:

- Expand big-O notation: $T(n) \leq cn + T(2n/3)$ for some c .
- Guess: $T(n) \leq 3cn$
- Induction step: assume that is true for all $i < n$.
- $T(n) \leq cn + 3c(2n/3) = cn + 2cn = 3cn \quad \square$

Suffix array: sorting suffixes

Another idea: Use a sort algorithm that's aware that the items being sorted are all suffixes of the same string

Original suffix array paper suggested an $O(m \log m)$ algorithm

Manber U, Myers G. "Suffix arrays: a new method for on-line string searches." *SIAM Journal on Computing* 22.5 (1993): 935-948.

Other popular $O(m \log m)$ algorithms have been suggested

Larsson NJ, Sadakane K. Faster suffix sorting. Technical Report LU-CS-TR: 99-214, LUNDFD6/(NFCS-3140)/1-43/(1999), Department of Computer Science, Lund University, Sweden, 1999.

More recently $O(m)$ algorithms have been demonstrated!

Kärkkäinen J, Sanders P. "Simple linear work suffix array construction." *Automata, Languages and Programming* (2003): 187-187.

Ko P, Aluru S. "Space efficient linear time construction of suffix arrays." *Combinatorial Pattern Matching*. Springer Berlin Heidelberg, 2003.

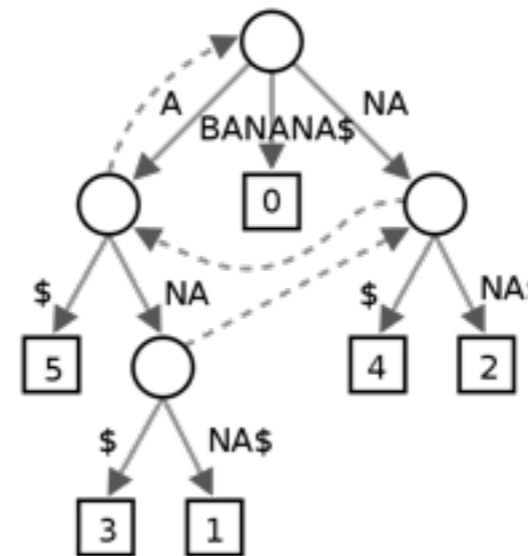
And there are comparable advances with respect to LCP1

Suffix array: summary

Suffix array gives us index that is:

(a) Just m integers, with $O(n \log m)$ worst-case query time, but close to $O(n + \log m)$ in practice

or **(b)** $3m$ integers, with $O(n + \log m)$ worst case



Suffix Tree

6	\$
5	A\$
3	ANA\$
1	ANANA\$
0	BANANA\$
4	NA\$
2	NANA\$

Suffix Array

(a) will often be preferable: index for entire human genome fits in ~12 GB instead of > 45 GB

Enhanced Suffix Arrays

Abouelhoda, Mohamed Ibrahim, Stefan Kurtz, and Enno Ohlebusch. "**Replacing suffix trees with enhanced suffix arrays.**" *Journal of Discrete Algorithms* 2.1 (2004): 53-86.

Can restore the **full** asymptotic efficiency of suffix trees with a small number of auxiliary tables.

Application	Enhanced suffix array					
	suftab $4n$ bytes	lcptab n bytes	childtab n bytes	suflink $2n$ bytes	S $n \log \Sigma $ bits	bwttab $n \log \Sigma $ bits
<i>esasupermax</i>	✓	✓				✓
<i>esamum</i>	✓	✓				✓
<i>esarep</i>	✓	✓				✓
Ziv–Lempel	✓	✓				
<i>esamatch</i>	✓	✓	✓		✓	
shortest unique sub.	✓	✓	✓			
<i>esams</i>	✓	✓	✓	✓	✓	

The operations that can be done optimally in an enhanced suffix array (esa), and the aux. tables required for them.

Many Suffix Array Variants

Compressed suffix arrays⁺ — require even less space

Compressed enhanced suffix arrays⁺ — strive for the best of both worlds and allow interesting query times like $O(n \log|\Sigma|+k)$ for finding k occurrences of a pattern, where $|\Sigma|$ is the size of the alphabet (independent of m).

⁺R. Grossi and J. S. Vitter, Compressed Suffix Arrays and Suffix Trees, with Applications to Text Indexing and String Matching, SIAM Journal on Computing, 35(2), 2005, 378-407

⁺Ohlebusch, Enno, and Simon Gog. "A compressed enhanced suffix array supporting fast string matching." String Processing and Information Retrieval. Springer Berlin Heidelberg, 2009. *