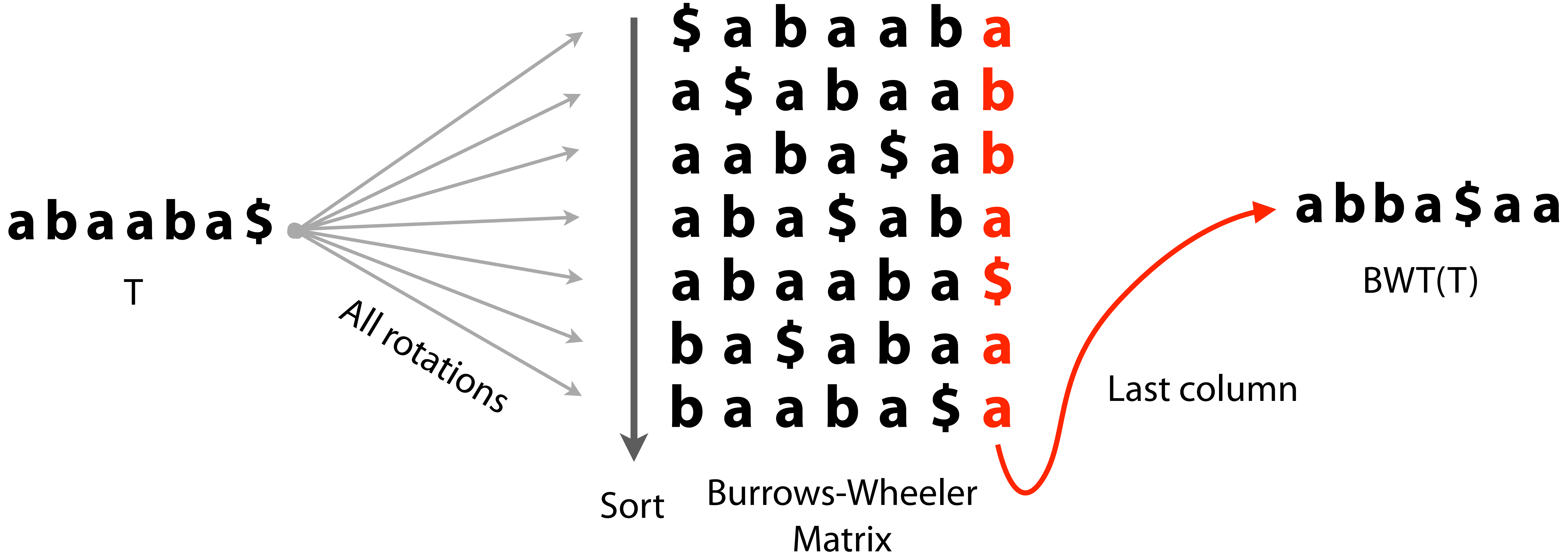
BWT & FM-index

All slides in this lecture **not** marked with “*” are courtesy of Ben Langmead (www.langmead-lab.org/teaching-materials).

Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

```
def rotations(t):
    """ Return list of rotations of input string t """
    tt = t * 2
    return [ tt[i:i+len(t)] for i in xrange(0, len(t)) ]
```

Make list of all rotations

```
def bwm(t):
    """ Return lexicographically sorted list of t’s rotations """
    return sorted(rotations(t))
```

Sort them

```
def bwtViaBwm(t):
    """ Given T, returns BWT(T) by way of the BWM """
    return ''.join(map(lambda x: x[-1], bwm(t)))
```

Take last column

```
>>> bwtViaBwm("Tomorrow_and_tomorrow_and_tomorrow$")
'w$wwdd__nnoooaattTmmmrrrrrrooo__ooo'

>>> bwtViaBwm("It_was_the_best_of_times_it_was_the_worst_of_times$")
's$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______'

>>> bwtViaBwm('in_the_jingle_jangle_morning_Ill_come_following_you$')
'u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_'
```

Burrows-Wheeler Transform

Characters of the BWT are sorted by their *right-context*

This lends additional structure to BWT(T), tending to make it more compressible

final char (L)	sorted rotations
a	n to decompress. It achieves compression
o	n to perform only comparisons to a depth
o	n transformation} This section describes
o	n transformation} We use the example and
o	n treats the right-hand side as the most
a	n tree for each 16 kbyte input block, enc
a	n tree in the output stream, then encodes
i	n turn, set $L[i]$ to be the
i	n turn, set $R[i]$ to the
o	n unusual data. Like the algorithm of Man
a	n use a single set of probabilities table
e	n using the positions of the suffixes in
i	n value at a given point in the vector $R
e	n we present modifications that improve t
e	n when the block size is quite large. Ho
i	n which codes that have not been seen in
i	n with ch appear in the {\em same order
i	n with ch. In our exam
o	n with Huffman or arithmetic coding. Bri
o	n with figures given by Bell~\cite{bell}.

Figure 1: Example of sorted rotations. Twenty consecutive rotations from the sorted list of rotations of a version of this paper are shown, together with the final character of each rotation.

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm.
Digital Equipment Corporation, Palo Alto, CA 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

BWM bears a resemblance to the suffix array

$	a	b	a	a	b	a
a	$	a	b	a	a	b
a	a	b	a	$	a	b
a	b	a	$	a	b	a
a	b	a	a	b	a	$
b	a	$	a	b	a	a
b	a	a	b	a	$	a

BWM(T)

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

SA(T)

Sort order is the same whether rows are rotations or suffixes

Burrows-Wheeler Transform

In fact, this gives us a new definition / way to construct BWT(T):

$$BWT[i] = \begin{cases} T[SA[i] - 1] & \text{if } SA[i] > 0 \\ \$ & \text{if } SA[i] = 0 \end{cases}$$

"BWT = characters just to the left of the suffixes in the suffix array"

$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a

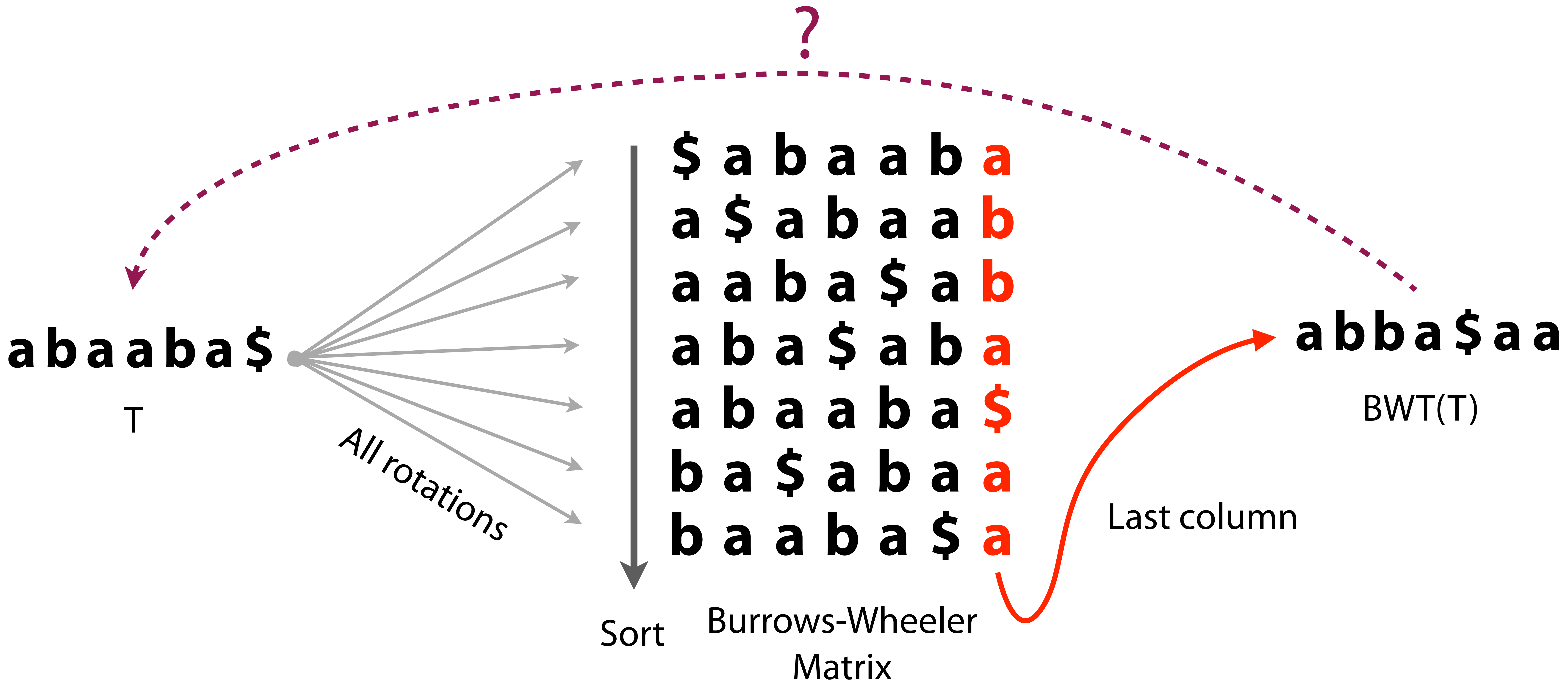
BWM(T)

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

SA(T)

Burrows-Wheeler Transform

How to reverse the BWT?



BWM has a key property called the *LF Mapping*...

Burrows-Wheeler Transform: T-ranking

Give each character in *T* a rank, equal to # times the character occurred previously in *T*. Call this the *T-ranking.*

$$\mathbf{a_0\ b_0\ a_1\ a_2\ b_1\ a_3\ \$}$$

Now let's re-write the BWM including ranks...

Note: we *do not* actually write this information in the text / BWM, we Are simply including it here to help us track "which" occurrences of each character in the BWM correspond to the occurrences in the text.

Burrows-Wheeler Transform

BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	\$	a_0	b_0
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
b_1	a_3	\$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	\$	a_0

Look at first and last columns, called *F* and *L*

And look at just the **a**s

as occur in the same order in *F* and *L*. As we look down columns, in both cases we see: a_3, a_1, a_2, a_0

Burrows-Wheeler Transform

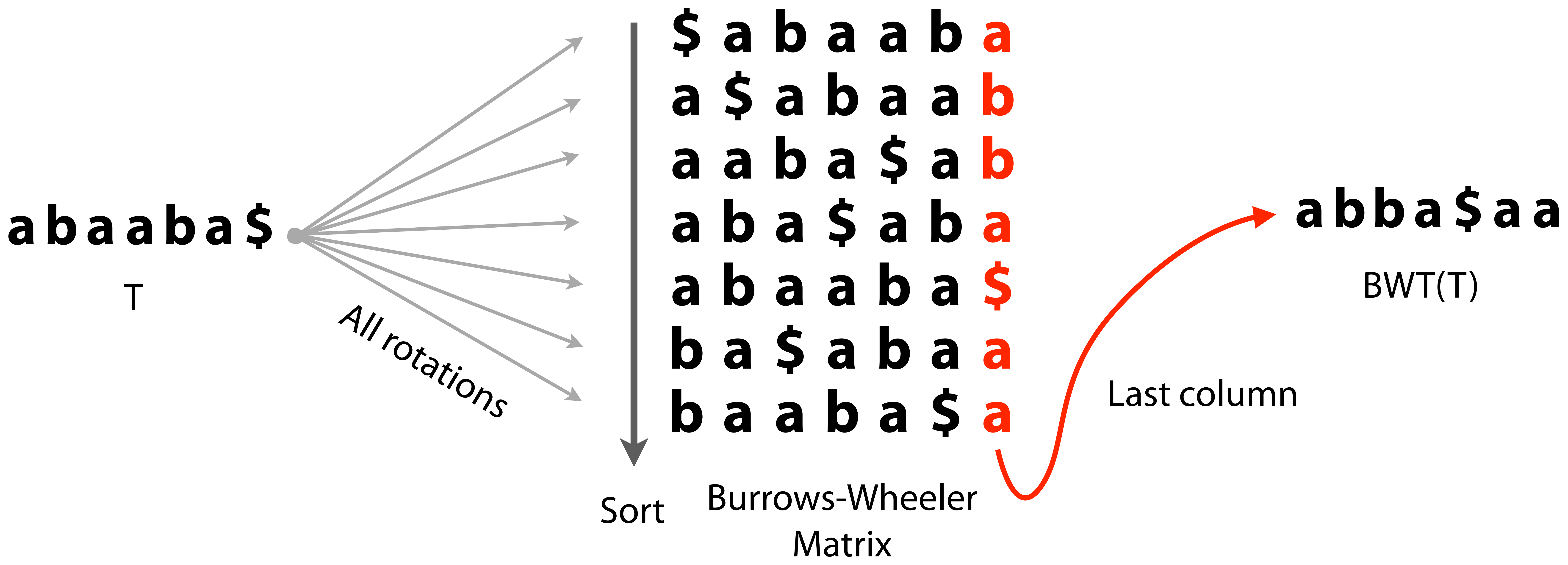
BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	$\mathbf{b_1}$
a_1	a_2	b_1	a_3	\$	a_0	$\mathbf{b_0}$
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
$\mathbf{b_1}$	a_3	\$	a_0	b_0	a_1	a_2
$\mathbf{b_0}$	a_1	a_2	b_1	a_3	\$	a_0

Same with **b**s: $\mathbf{b_1}$, $\mathbf{b_0}$

Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm.
Digital Equipment Corporation, Palo Alto, CA 1994, Technical Report 124; 1994

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						L
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	\$	a_0	b_0
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
b_1	a_3	\$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	\$	a_0

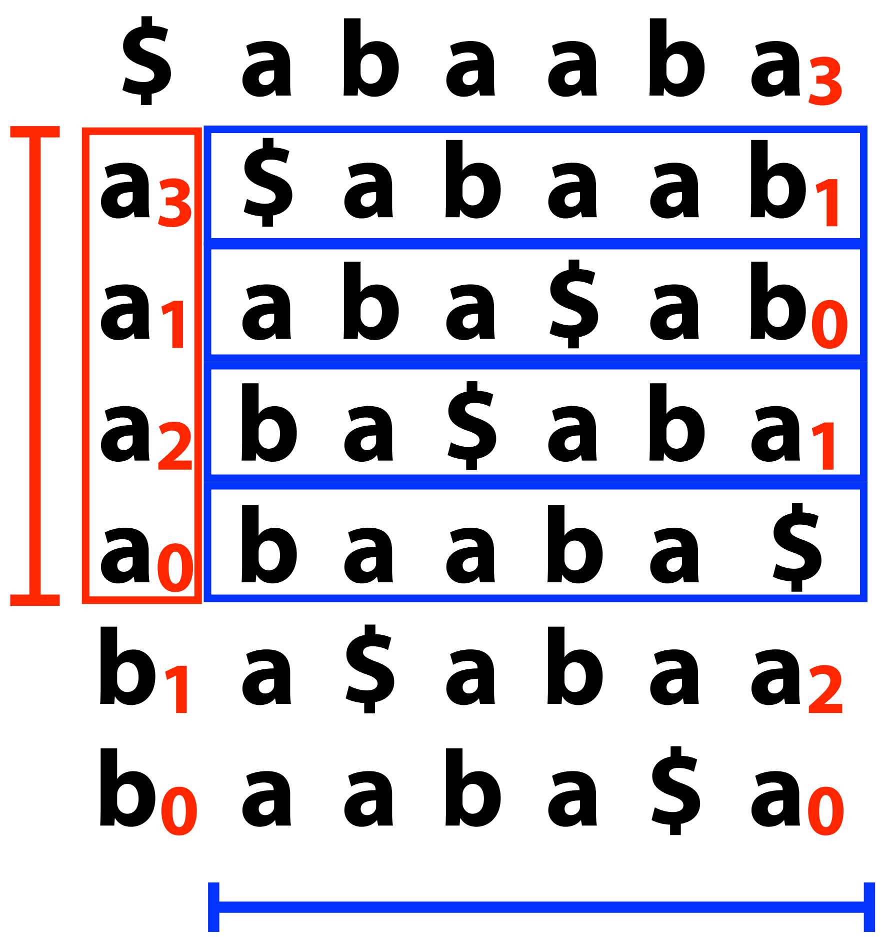
LF Mapping: The i^{th} occurrence of a character c in L and the i^{th} occurrence of c in F correspond to the *same* occurrence in T

However we rank occurrences of c, ranks appear in the same order in F and L

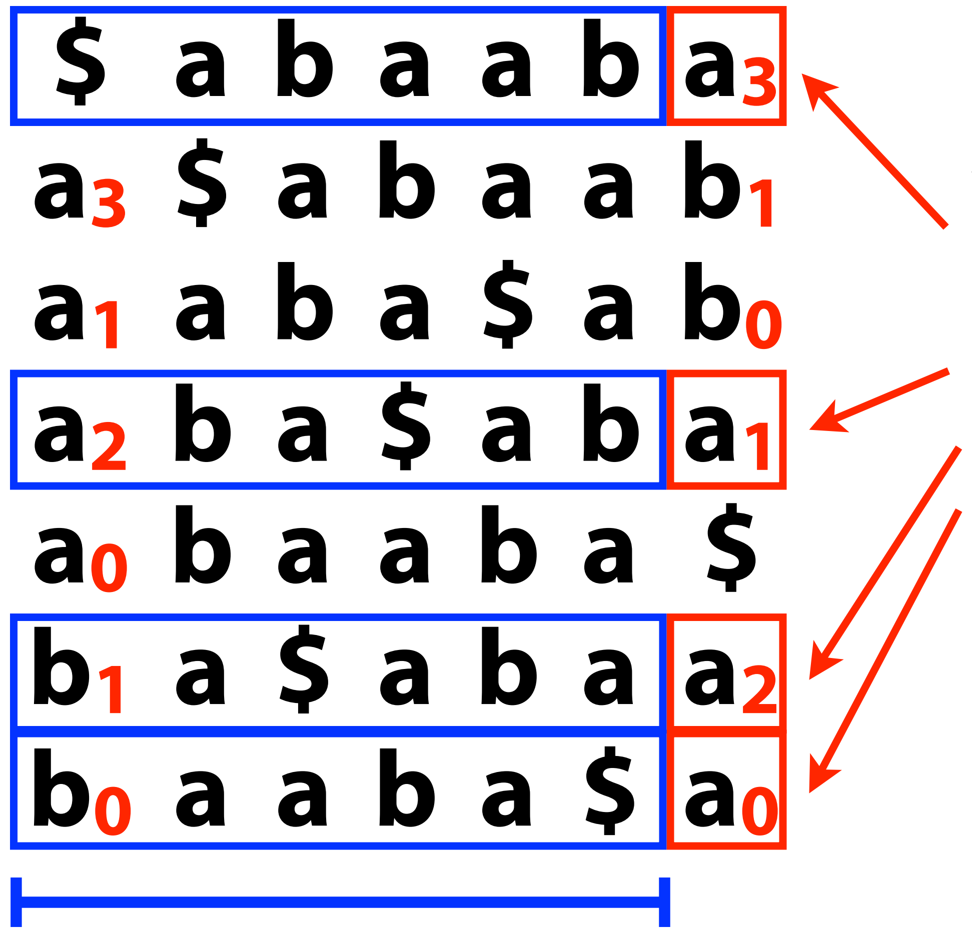
Burrows-Wheeler Transform: LF Mapping

Why does the LF Mapping hold?

Why are these **a**s in this order relative to each other?



They're sorted by right-context



They're sorted by right-context

Why are these **a**s in this order relative to each other?

Occurrences of *c* in *F* are sorted by right-context. Same for *L*!

Whatever ranking we give to characters in *T*, rank orders in *F* and *L* will match

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	$	a_0	b_0
a_2	b_1	a_3	$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	$
b_1	a_3	$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	$	a_0

We'd like a different ranking so that for a given character, ranks are in ascending order as we look down the F / L columns...

Burrows-Wheeler Transform: LF Mapping

BWM with B-ranking:

F						L
\$	a_3	b_1	a_1	a_2	b_0	$\mathbf{a_0}$
$\mathbf{a_0}$	\$	a_3	b_1	a_1	a_2	$\mathbf{b_0}$
$\mathbf{a_1}$	a_2	b_0	a_3	\$	a_3	$\mathbf{b_1}$
$\mathbf{a_2}$	b_0	a_0	\$	a_3	b_1	$\mathbf{a_1}$
$\mathbf{a_3}$	b_1	a_1	a_2	b_0	a_0	**\$**
$\mathbf{b_0}$	a_0	\$	a_3	b_1	a_1	$\mathbf{a_2}$
$\mathbf{b_1}$	a_1	a_2	b_0	a_0	\$	$\mathbf{a_3}$

Ascending rank

F now has very simple structure: a **\$**, a block of **a**s *with ascending ranks*, a block of **b**s *with ascending ranks*

Burrows-Wheeler Transform

	F	L
	$\$$	a_0
	a_0	b_0
	a_1	b_1 ←
	a_2	a_1
	a_3	$\$$
	b_0	a_2
row 6 →	b_1	a_3

Which BWM row *begins* with $\mathbf{b}_1$?

Skip row starting with $\$$ (1 row)

Skip rows starting with **a** (4 rows)

Skip row starting with $\mathbf{b}_0$ (1 row)

Answer: row 6

Burrows-Wheeler Transform

Say *T* has 300 **A**s, 400 **C**s, 250 **G**s and 700 **T**s and **\$** < **A** < **C** < **G** < **T**

Which BWM row (0-based) begins with $\mathbf{G}_{100}$? (Ranks are B-ranks.)

Skip row starting with **\$** (1 row)

Skip rows starting with **A** (300 rows)

Skip rows starting with **C** (400 rows)

Skip first 100 rows starting with **G** (100 rows)

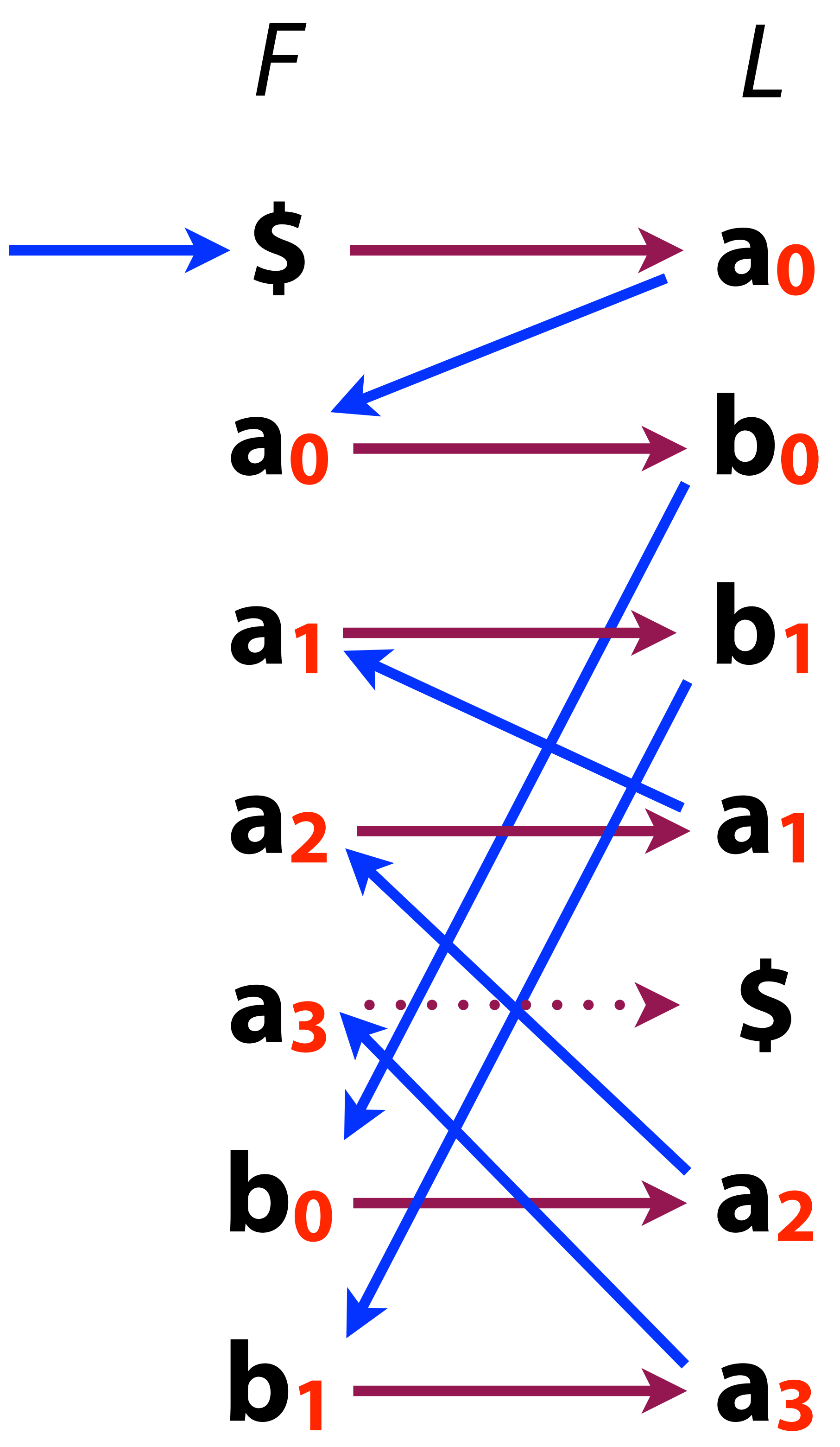
Answer: row 1 + 300 + 400 + 100 = **row 801**

Burrows-Wheeler Transform: reversing

Reverse BWT(T) starting at right-hand-side of *T* and moving left

Start in first row. *F* must have **$**. *L* contains character just prior to **$**: $\mathbf{a}_0$

$\mathbf{a}_0$: LF Mapping says this is same occurrence of **a** as first **a** in *F*. Jump to row *beginning* with $\mathbf{a}_0$. *L* contains character just prior to $\mathbf{a}_0$: $\mathbf{b}_0$.



Repeat for $\mathbf{b}_0$, get $\mathbf{a}_2$

Repeat for $\mathbf{a}_2$, get $\mathbf{a}_1$

Repeat for $\mathbf{a}_1$, get $\mathbf{b}_1$

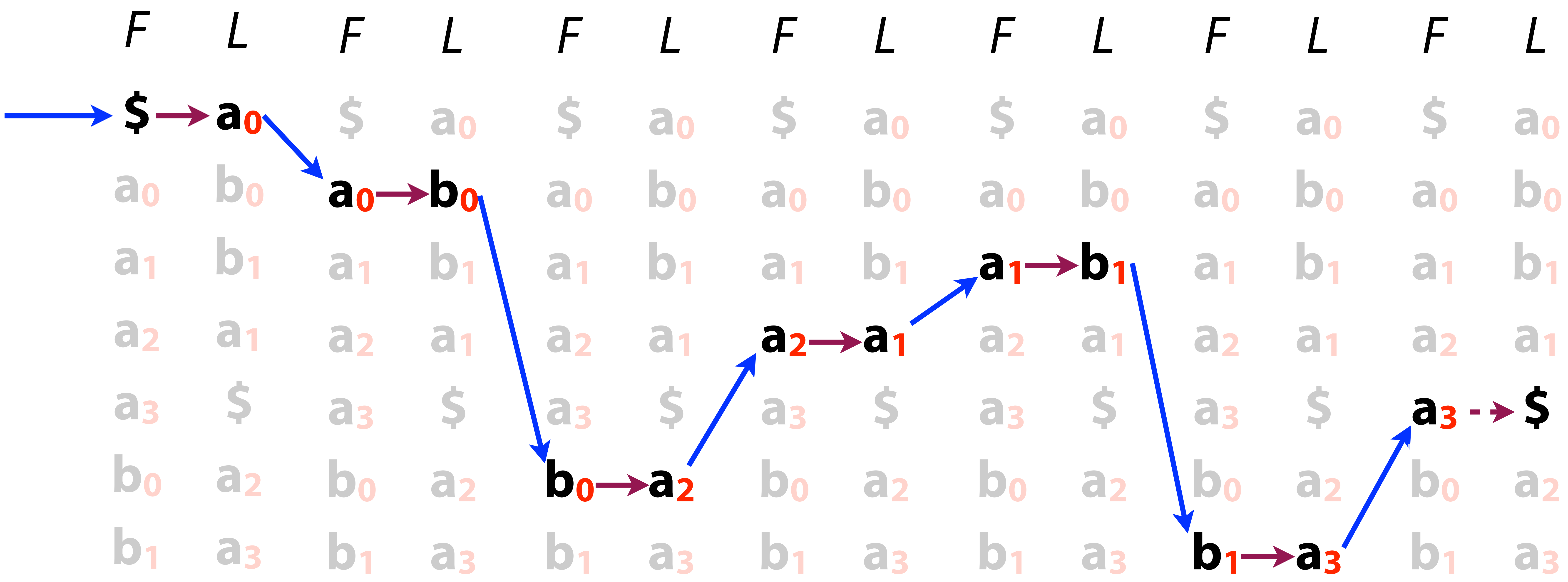
Repeat for $\mathbf{b}_1$, get $\mathbf{a}_3$

Repeat for $\mathbf{a}_3$, get **$**, done

Reverse of chars we visited = $\mathbf{a}_3\ \mathbf{b}_1\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_0\ \mathbf{a}_0\ \$ = T$

Burrows-Wheeler Transform: reversing

Another way to visualize reversing BWT(T):



T: $a_3\ b_1\ a_1\ a_2\ b_0\ a_0\ \$$

Burrows-Wheeler Transform: reversing

```
>>> reverseBwt("w$wwdd__nnoooaattTmmmrrrrrrooo__ooo")
'Tomorrow_and_tomorrow_and_tomorrow$'

>>> reverseBwt("s$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______")
'It_was_the_best_of_times_it_was_the_worst_of_times$'

>>> reverseBwt("u_gleeeengj_mlhl_nnnnt$nwj__lggIolo_iiiiarfcmylo_oo_")
'in_the_jingle_jangle_morning_Ill_come_following_you$'
```

ranks list is *m* integers long! We'll fix later.

```
def reverseBwt(bw):
    ''' Make T from BWT(T) '''
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0 # start in first row
    t = '$' # start with rightmost character
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t # prepend to answer
        # jump to row that starts with c of same rank
        rowi = first[c][0] + ranks[rowi]
    return t
```

Burrows-Wheeler Transform

We've seen how BWT is useful for compression:

- Sorts characters by right-context, making a more compressible string

And how it's reversible:

- Repeated applications of LF Mapping, recreating *T* from right to left

How is it used as an index?

FM Index

FM Index: an index combining the BWT *with a few small auxilliary data structures*

"FM" supposedly stands for "Full-text Minute-space."
(But inventors are named Ferragina and Manzini)

Core of index consists of *F* and *L* from BWM:

F can be represented very simply
(1 integer per alphabet character)

And *L* is compressible

Potentially very space-economical!

F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

Not stored in index

Paolo Ferragina, and Giovanni Manzini. "Opportunistic data structures with applications." *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*. IEEE, 2000.

FM Index: querying

Though BWM is related to suffix array, we can't query it the same way

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

We don't have these columns; binary search isn't possible

FM Index: querying

Look for range of rows of BWM(T) with *P* as prefix

Do this for *P*'s shortest suffix, then extend to successively longer suffixes until range becomes empty or we've exhausted *P*

P = **aba**

Easy to find all the rows beginning with **a**, thanks to *F*'s simple structure

F						*L*
\$	a	b	a	a	b	$\mathbf{a}_3$
$\mathbf{a}_0$	\$	a	b	a	a	$\mathbf{b}_1$
$\mathbf{a}_1$	a	b	a	\$	a	$\mathbf{b}_0$
$\mathbf{a}_2$	b	a	\$	a	b	$\mathbf{a}_1$
$\mathbf{a}_3$	b	a	a	b	a	**\$**
$\mathbf{b}_0$	a	\$	a	b	a	$\mathbf{a}_2$
$\mathbf{b}_1$	a	a	b	a	\$	$\mathbf{a}_0$

FM Index: querying

We have rows beginning with **a**, now we seek rows beginning with **ba**

$P =$ **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

← Look at those rows in *L*.

b_0, b_1 are **b**s occuring just to left.

Use LF Mapping. Let new range delimit those **b**s →

$P =$ **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

Now we have the rows with prefix **ba**

FM Index: querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**

$P =$ **aba**

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

← a_2, a_3 occur just to left.

Use LF Mapping →

$P =$ **aba**

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Now we have the rows with prefix **aba**

FM Index: querying

$P =$ **aba**

Now we have the same range, [3, 5), we would have got from querying suffix array

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

[3, 5)

Where are these?

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

[3, 5)

Unlike suffix array, we don't immediately know *where* the matches are in T...

FM Index: querying

When *P* does not occur in *T*, we will eventually fail to find the next character in *L*:

$P =$ **bba**

F					*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Rows with **ba** prefix

← No **b**s!

FM Index: querying

If we *scan* characters in the last column, that can be very slow, $O(m)$

$P =$ **aba**

F						L
$	a	b	a	a	b	a_3
a_0	$	a	b	a	a	b_1
a_1	a	b	a	$	a	b_0
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_0

Scan, looking for **b**s

FM Index: lingering issues

(1) Scanning for preceding character is slow

$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

$O(m)$ scan

(2) Storing ranks takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

(3) Need way to find where matches occur in *T*:

Where?

$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

FM Index: fast rank calculations

Is there an O(1) way to determine which **b**s precede the **a**s in our range?

F						*L*
$	a	b	a	a	b	$\mathbf{a}_0$
$\mathbf{a}_0$	$	a	b	a	a	$\mathbf{b}_0$
$\mathbf{a}_1$	a	b	a	$	a	$\mathbf{b}_1$
$\mathbf{a}_2$	b	a	$	a	b	$\mathbf{a}_1$
$\mathbf{a}_3$	b	a	a	b	a	**$**
$\mathbf{b}_0$	a	$	a	b	a	$\mathbf{a}_2$
$\mathbf{b}_1$	a	a	b	a	$	$\mathbf{a}_3$

Occ(c, k) = # of of c in the first k characters of BWT(S), aka the LF mapping.

Tally — also referred to as **Occ**(c, k)

Idea: pre-calculate # **a**s, **b**s in *L* up to every row:

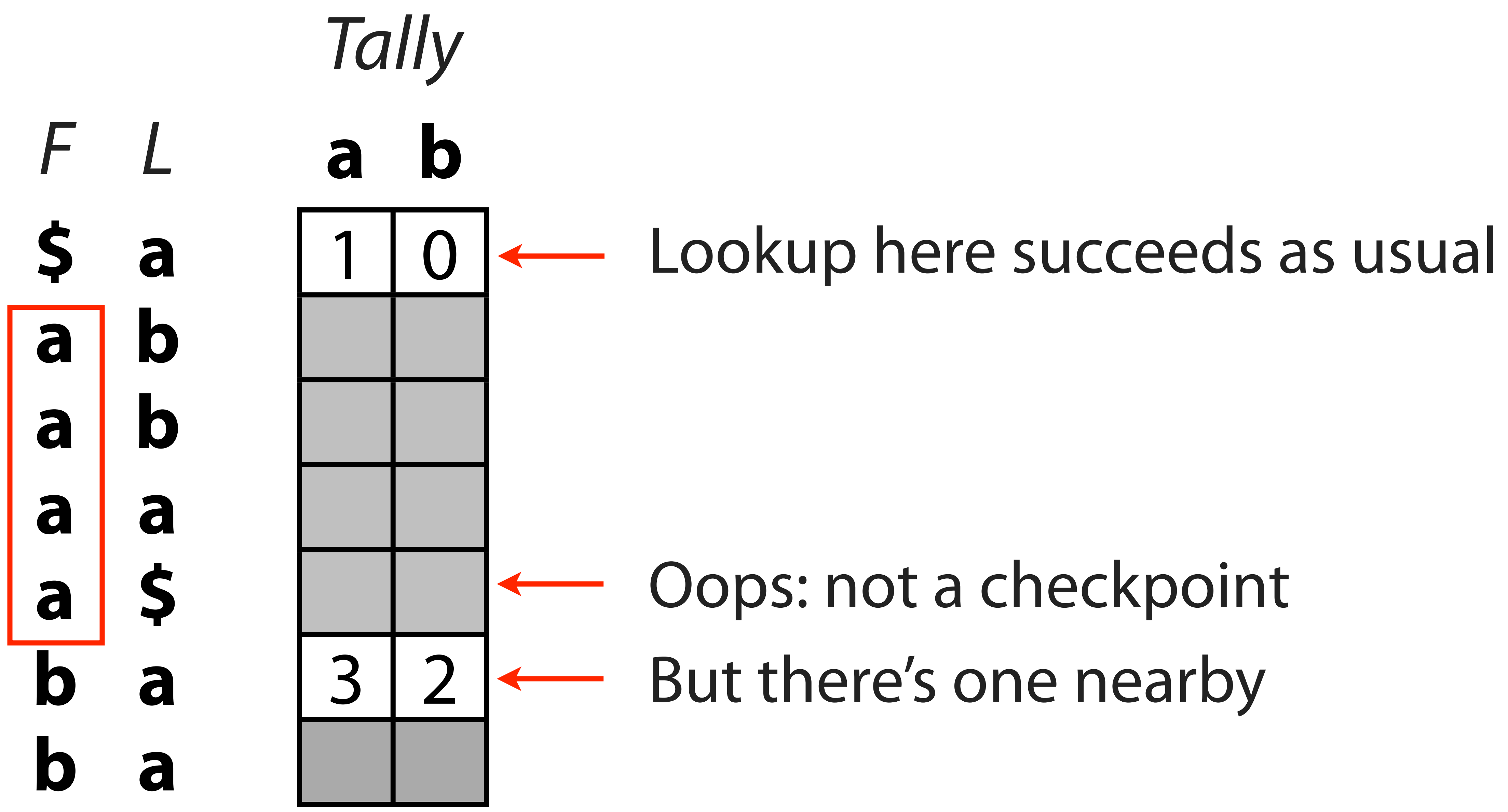
F	*L*	**a**	**b**
$	**a**	1	0
a	**b**	1	1
a	**b**	1	2
a	**a**	2	2
a	**$**	2	2
b	**a**	3	2
b	**a**	4	2

We infer $\mathbf{b}_0$ and $\mathbf{b}_1$ appear in *L* in this range

O(1) time, but requires $m \times |\Sigma|$ integers

FM Index: fast rank calculations

Another idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row. Call pre-calculated rows *checkpoints*.



To resolve a lookup for character *c* in non-checkpoint row, scan along *L* until we get to nearest checkpoint. Use tally at the checkpoint, *adjusted for # of cs we saw along the way*.

FM Index: fast rank calculations

What's my rank?

482 + 2 - 1 = 483

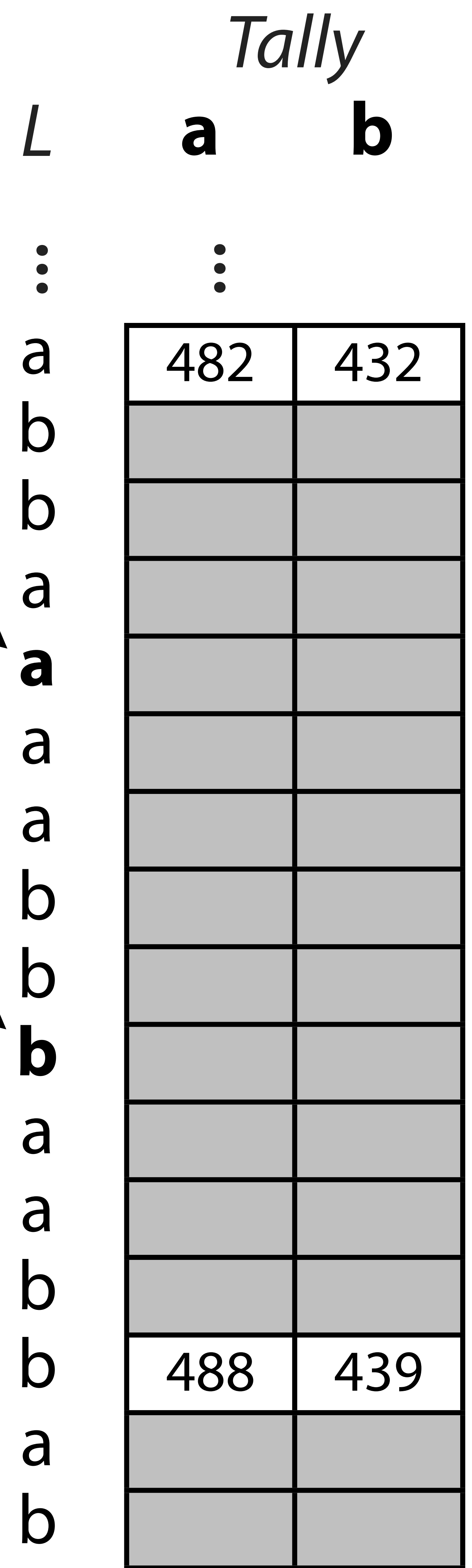
Checkpoint

as along the way

tally -> rank

What's my rank?

439 - 2 - 1 = 436



Assuming checkpoints are spaced $O(1)$ distance apart, lookups are $O(1)$

FM Index: fast rank calculations

This can also be accomplished using **bit-vector rank** operations. We store one bit-vector for each character of Σ, placing a 1 where this character occurs and a 0 everywhere else:

		Tally	
F	*L*	**a**	**b**
$	**a**	1	0
a	**b**	0	1
a	**b**	0	1
a	**a**	1	0
a	**$**	0	0
b	**a**	1	0
b	**a**	1	0

the operation **rank(x, i)** returns the total number of 1's in a bit-vector up to (and including) index i. **rank(x,i)** is a *constant-time operation*

rank(a,3) = 2

rank(a,5) = 3

rank(b,5) = 2

To resolve the rank for a given character **c** at a given index **i**, we simply issue a **rank(c,i)** query. This is a practically-fast constant-time operation, but we need to keep around Σ bit-vectors, each of o(m) bits.

FM Index: a few problems

Solved! At the expense of adding checkpoints ($O(m)$ integers) to index.

(1)

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

This scan is $O(m)$ work

With checkpoints it's $O(1)$

(2) Ranking takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

With checkpoints, we greatly reduce # integers needed for ranks

But it's still $O(m)$ space - there's literature on how to improve this space bound

FM Index: a few problems

Not yet solved: **(3)** Need a way to find where these occurrences are in *T*:

\$	a	b	a	a	b	**a_0**
a_0	\$	a	b	a	a	**b_0**
a_1	a	b	a	\$	a	**b_1**
a_2	b	a	\$	a	b	**a_1**
a_3	b	a	a	b	a	**\$**
b_0	a	\$	a	b	a	**a_2**
b_1	a	a	b	a	\$	**a_3**

If suffix array were part of index, we could simply look up the offsets

F						*L*
\$	a	b	a	a	b	**a**
a	\$	a	b	a	a	**b**
a	a	b	a	\$	a	**b**
a	b	a	\$	a	b	**a**
a	b	a	a	b	a	**\$**
b	a	\$	a	b	a	**a**
b	a	a	b	a	\$	**a**

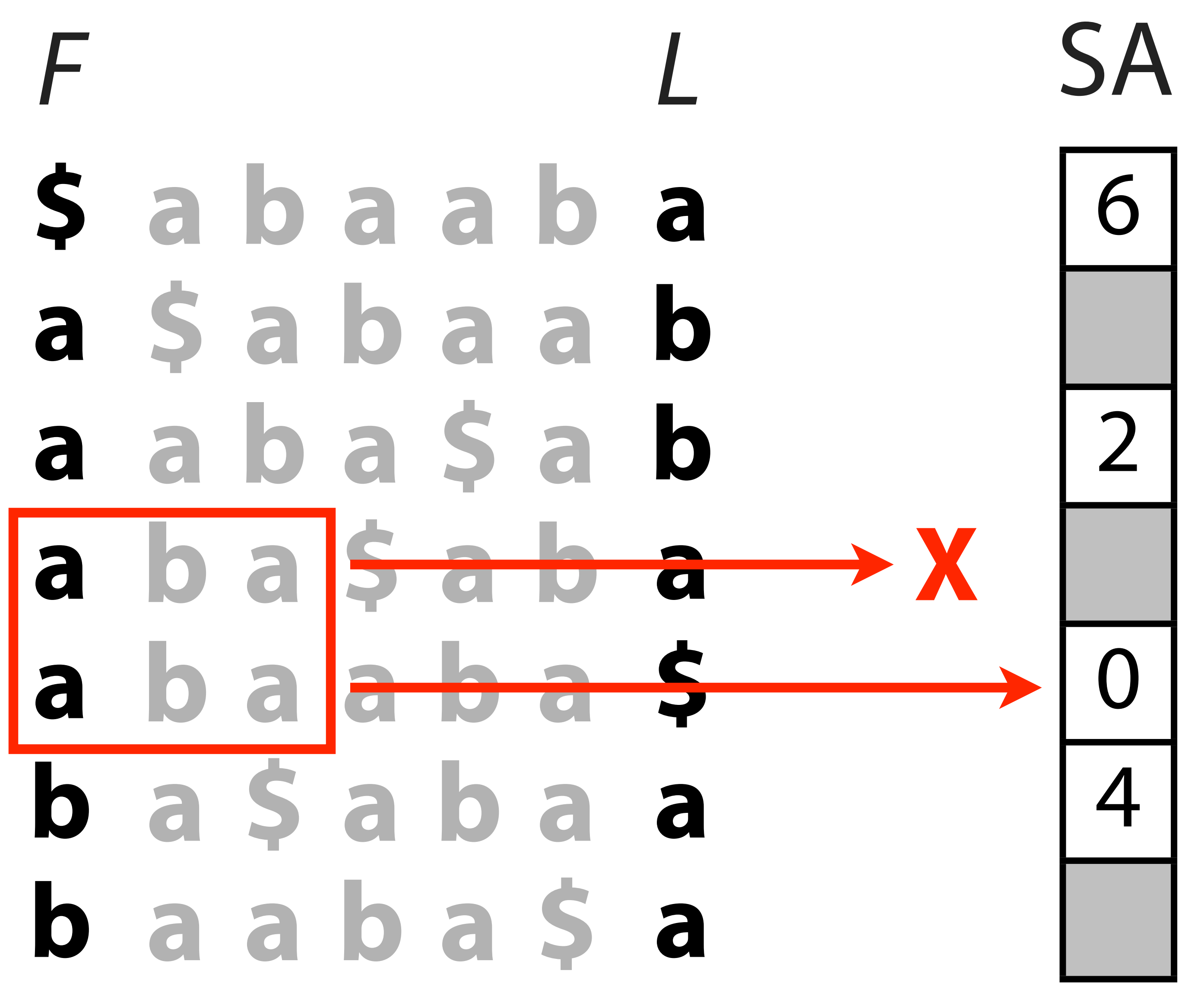
Offsets: 0, 3

SA	
6	**\$**
5	**a \$**
2	**a a b a \$**
3	**a b a \$**
0	**a b a a b a \$**
4	**b a \$**
1	**b a a b a \$**

But SA requires *m* integers

FM Index: resolving offsets

Idea: store some, but not all, entries of the suffix array



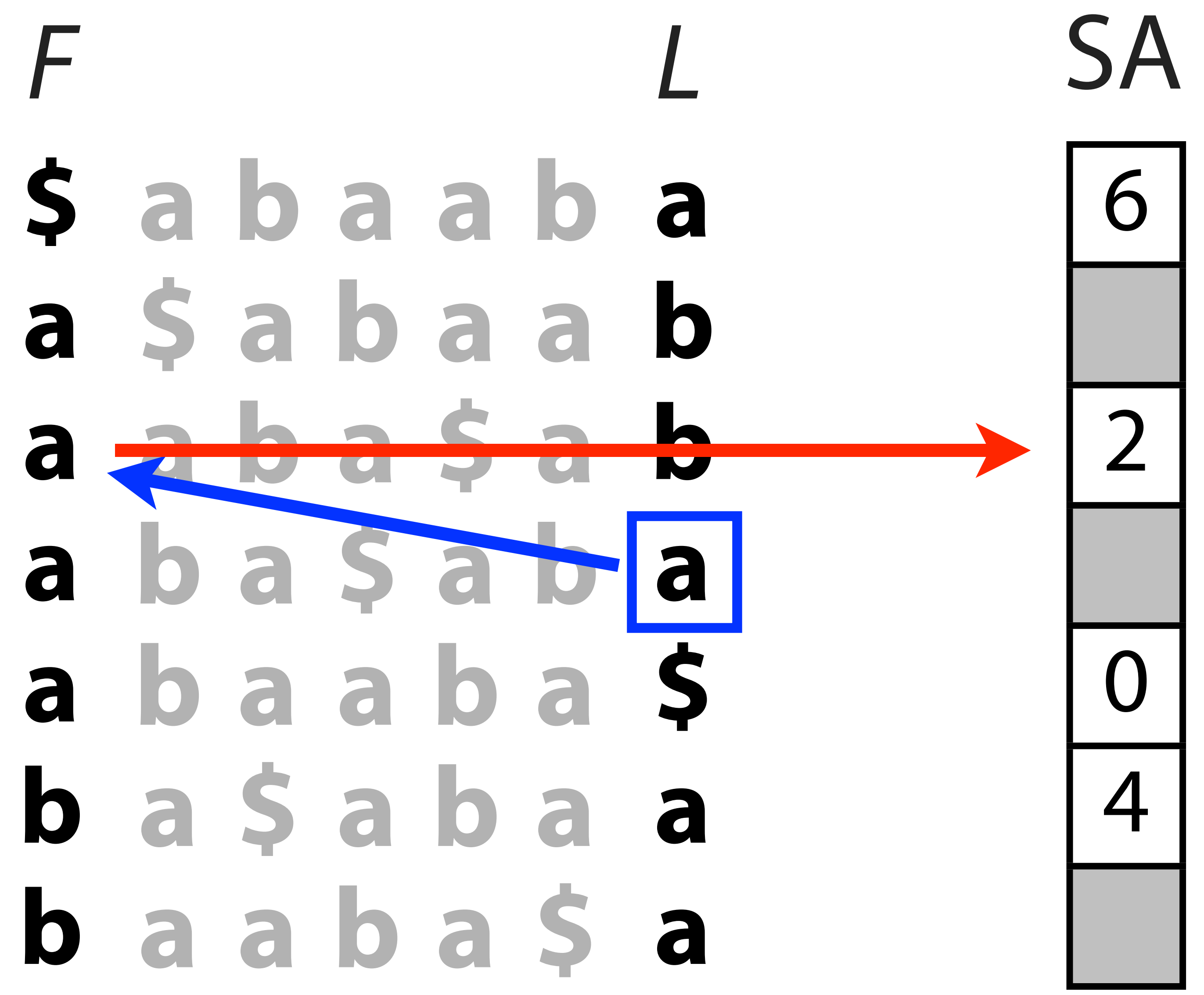
Lookup for row 4 succeeds - we kept that entry of SA

Lookup for row 3 fails - we discarded that entry of SA

FM Index: resolving offsets

But LF Mapping tells us that the **a** at the end of row 3 corresponds to...

...the **a** at the begining of row 2



And row 2 has a suffix array value = 2

So row 3 has suffix array value = 3 = 2 (row 2's SA val) + 1 (# steps to row 2)

If saved SA values are O(1) positions apart in *T*, resolving offset is O(1) time

FM Index: problems solved

Solved!

At the expense of adding some SA values ($O(m)$ integers) to index

Call this the "SA sample"

(3) Need a way to find where these occurrences are in *T*:

$	a	b	a	a	b	**a_0**
a_0	$	a	b	a	a	**b_0**
a_1	a	b	a	$	a	**b_1**
a_2	b	a	$	a	b	**a_1**
a_3	b	a	a	b	a	**$**
b_0	a	$	a	b	a	**a_2**
b_1	a	a	b	a	$	**a_3**

With SA sample we can do this in *O*(1) time per occurrence

FM Index: small memory footprint

Components of the FM Index:

First column (F):	$\sim \lvert \Sigma \rvert$ integers
Last column (L):	m characters
SA sample:	$m \cdot a$ integers, where a is fraction of rows kept
Checkpoints:	$m \times \lvert \Sigma \rvert \cdot b$ integers, where b is fraction of rows checkpointed

Example: DNA alphabet (2 bits per nucleotide), T = human genome, $a = 1/32$, $b = 1/128$

First column (F):	16 bytes
Last column (L):	2 bits * 3 billion chars = 750 MB
SA sample:	3 billion chars * 4 bytes/char / 32 = ~ 400 MB
Checkpoints:	3 billion * 4 bytes/char * 4 char / 120 = ~400MB

Total ~ 1.5 GB

Computing BWT in O(n) time

- Easy O(n^2 log n)-time algorithm to compute the BWT (create and sort the BWT matrix explicitly).

- Several direct O(n)-time algorithms for BWT.
These are space efficient. (Bowtie e.g. uses [1])

- Also can use suffix arrays or trees:

  Compute the suffix array, use correspondence between suffix array and BWT to output the BWT.

  O(n)-time and O(n)-space, but the constants are large.

[1] Kärkkäinen, Juha. "Fast BWT in small space by blockwise suffix sorting." *Theoretical Computer Science* 387.3 (2007): 249-257.

*slide courtesy of Carl Kingsford

Bonus material (not on exams … but cool!)

Actual FM-Index Built on Compressed String

Ferragina, Paolo, and Giovanni Manzini. "Opportunistic data structures with applications." Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on. IEEE, 2000.

Data structure has "space occupancy that is a *function of the entropy* of the underlying data set"

Stores text T[1,u] in $O(H_k(T)) + o(1)$ bits for $k \geq 0$ where $H_k(T)$ is the kith order empirical entropy of the text — **sub-linear** for a compressible string

Theorem 1 *Let Z denote the output of the algorithm* BW_RLX *on input $T[1,u]$. The number of occurrences of a pattern $P[1,p]$ in $T[1,u]$ can be computed in $O(p)$ time on a RAM. The space occupancy is $|Z| + O\left(\frac{u}{\log u} \log\log u\right)$ bits in the worst case.* ■

Theorem 2 *A text $T[1,u]$ can be preprocessed in $O(u)$ time so that all the occ occurrences of a pattern $P[1,p]$ in T can be listed in $O(p + occ\log^2 u)$ time on a RAM. The space occupancy is bounded by $5H_k(T) + O(\frac{\log\log u}{\log u})$ bits per input symbol in the worst case, for any fixed $k \geq 0$.* ■

Theorem 3 *A text $T[1,u]$ can be indexed so that all the occ occurrences of a pattern $P[1,p]$ in T can be listed in $O(p + occ\log^{\epsilon} u)$ time on a RAM. The space occupancy is $O(H_k(T) + \frac{\log\log u}{\log^{\epsilon} u})$ bits per input symbol in the worst case, for any fixed $k \geq 0$.* ■

Compressing BWT Strings

Lots of possible compression schemes will benefit from preprocessing with BWT (since it tends to group runs of the same letters together).

One good scheme proposed by Ferragina & Manzini:

replace runs of 0s with
the count of 0s

PrefixCode(rle(MTF(BWT(S))))

Huffman code that
uses more bits for rare
symbols

*slide courtesy of Carl Kingsford

Move-To-Front Coding

To encode a letter, use its index in the current list, and then move it to the front of the list.

Σ	do$oodwg
$dgow	1
d$gow	13
od$gw	132
$odgw	1321
o$dgw	13210
o$dgw	132102
do$gw	1321024
wdo$g	13210244 = MTF(do$oodwg)

List with all letters from the allowed alphabet

Benefits:

- Runs of the same letter will lead to runs of 0s.
- Common letters get small numbers, while rare letters get big numbers.

*slide courtesy of Carl Kingsford

Move-To-Front Decoding

To encode a letter, use its index in the current list, and then move it to the front of the list.

List with all letters from the allowed alphabet

Σ		
$dgow	13210244	d
d$gow	~~1~~3210244	do
od$gw	~~13~~210244	do$
$odgw	~~132~~10244	do$o
o$dgw	~~1321~~0244	do$oo
o$dgw	~~13210~~244	do$oog
do$gw	~~132102~~44	do$oogw
wdo$g	~~1321024~~4	do$oogwg

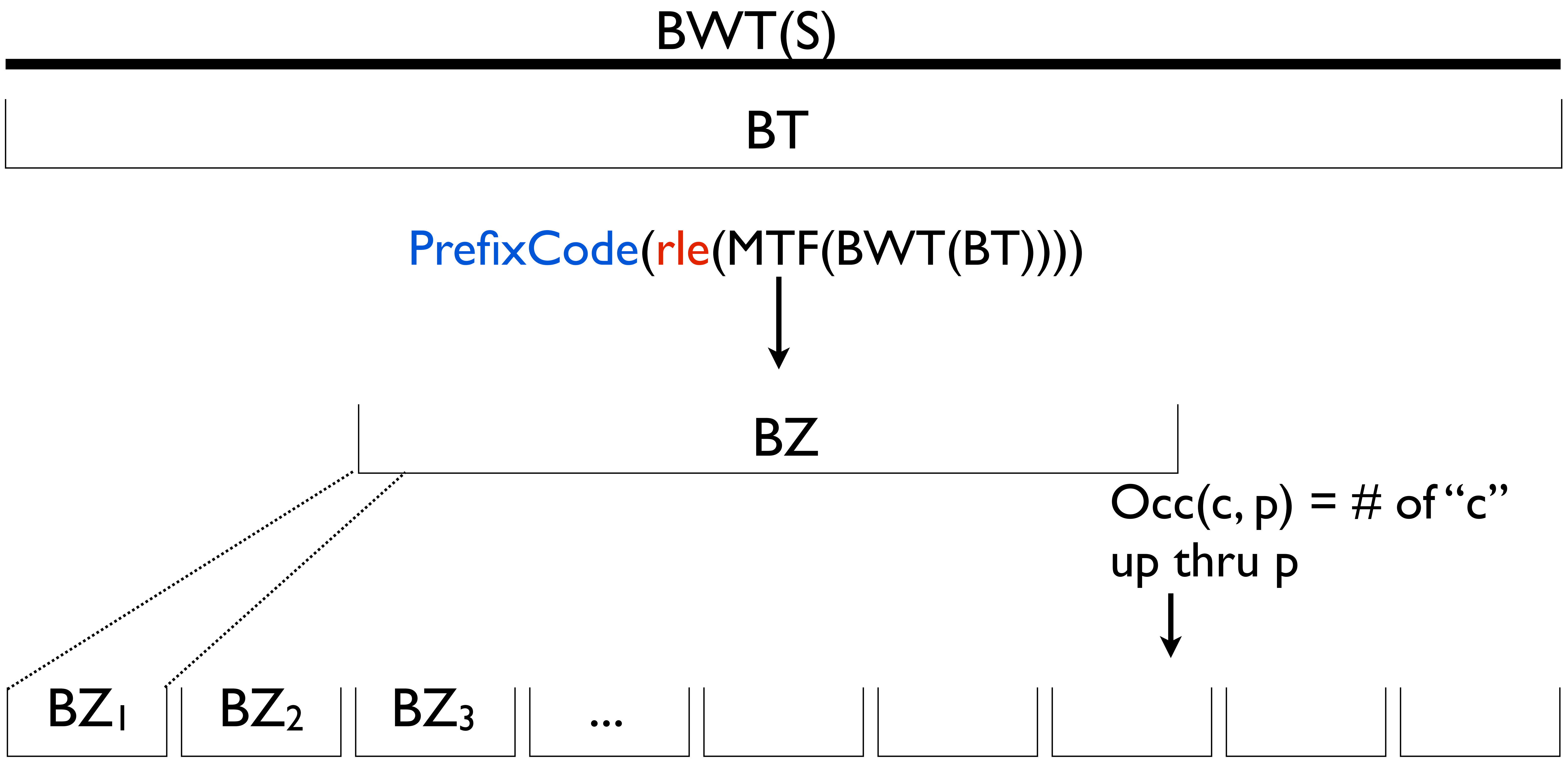
Benefits:

- Runs of the same letter will lead to runs of 0s.
- Common letters get small numbers, while rare letters get big numbers.

*slide courtesy of Carl Kingsford

Computing Occ in Compressed String

Break BWT(S) into blocks of length L (we will decide on a value for L later):



Assumes every run of 0s is contained in a block [just for ease of explanation].

We will store some extra info for each block (and some groups of blocks) to compute Occ(c, p) quickly.

*slide courtesy of Carl Kingsford

Extra Info to Compute Occ

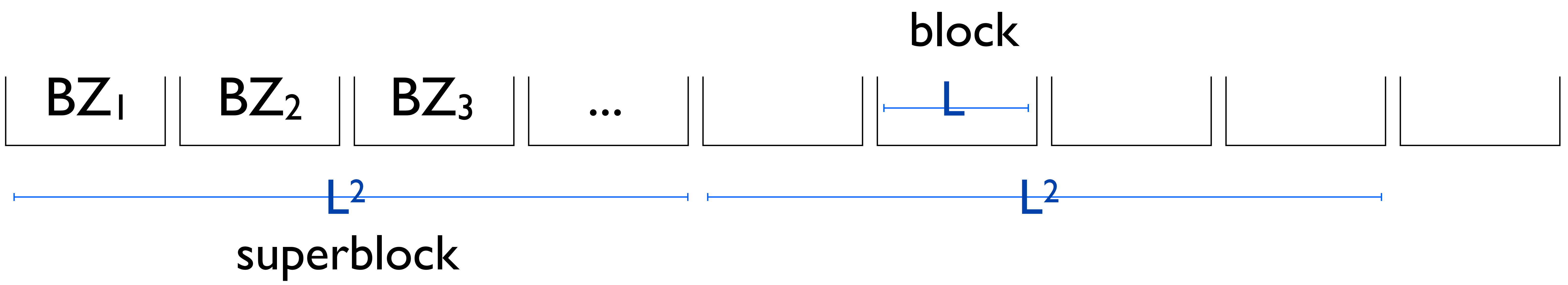
u = compressed length
Choose $L = O(\log u)$

u/L blocks, each array is $|\Sigma| \log L$ space

$\Longrightarrow$

$\frac{u}{L} \log L = \frac{u}{\log u} \log \log u$ total space.

block: store $|\Sigma|$-long array giving # of occurrences of each character up thru and including this block *since the end of the last super block.*



superblock: store $|\Sigma|$-long array giving # of occurrences of each character up thru *and including* this superblock

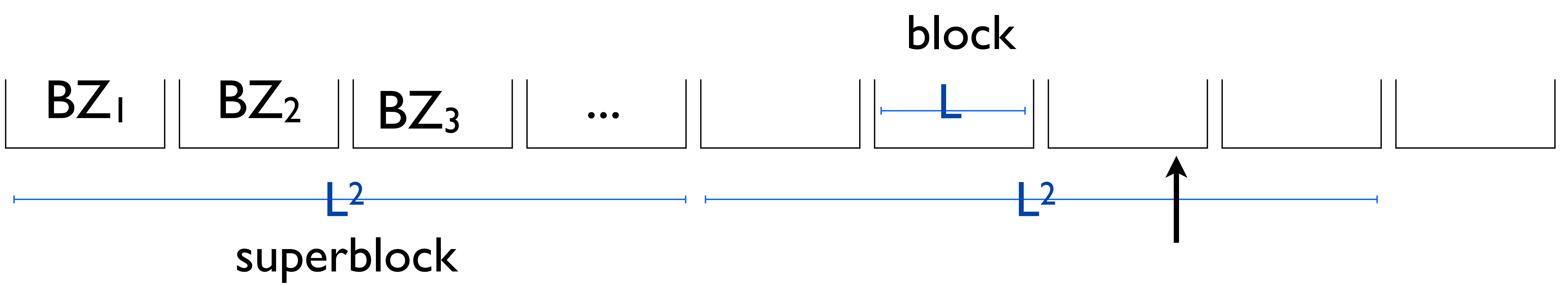
u/L^2 superblocks, each array is $|\Sigma| \log u$ long

$\Longrightarrow \frac{u}{(\log u)^2} \log u = \frac{u}{\log u}$ total space.

*slide courtesy of Carl Kingsford

Extra Info to Compute Occc

u = compressed length
Choose $L = O(\log u)$



Occ(c, p) = # of "c" up thru p:

sum value at last superblock, value at end of previous block, but then need to handle *this block.*

Store an array: M[c, k, BZ_i, MTF_i] = # of occurrences of c through the kth letter of a block of <u>type</u> (BZ_i, MTF_i).

\# of possible bit patterns of length log(u)

Size: $O(|\Sigma| L 2^L |\Sigma|) = O(L2^{L'}) = O(u^c \log u)$ for $c < 1$ (since the string is compressed)

*slide courtesy of Carl Kingsford